About air-shower universality

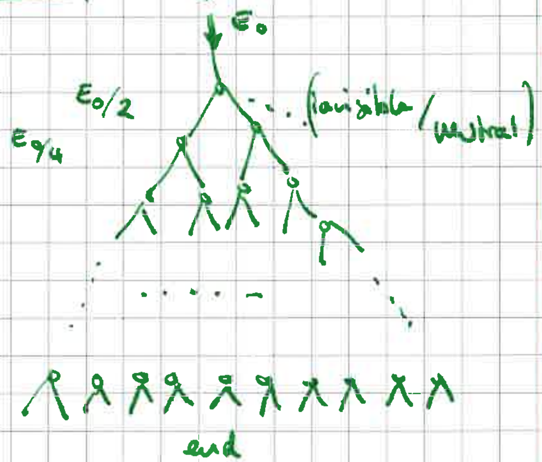
"all air-showers kind of look alike around the shower maximum"

~~###~~
the (electromagnetic) shower is in equilibrium at the shower maximum; all information about the state before the equilibrium is lost

BOTH self similarity

two models of air showers:

Heitler/Matthews



(+)
simple
tells you that

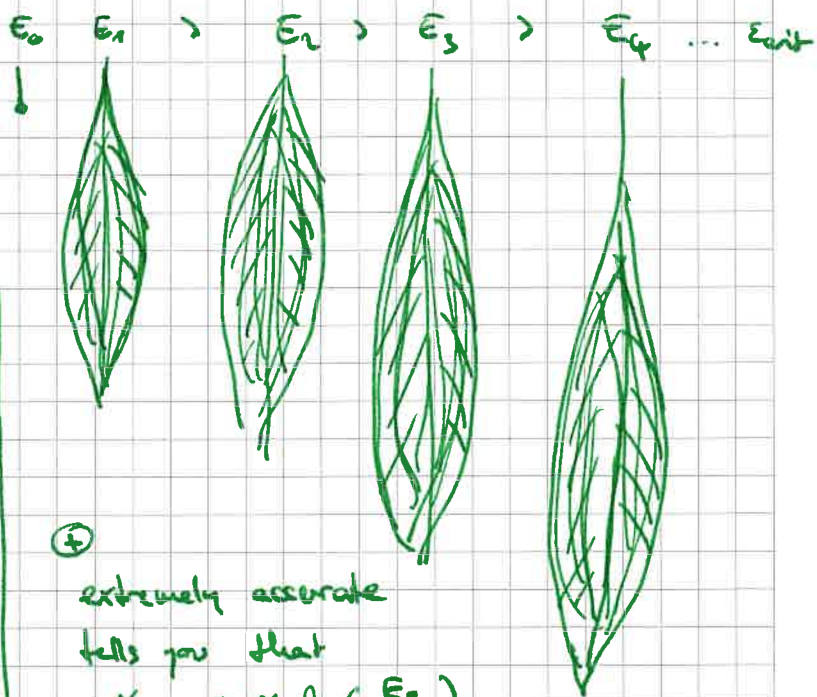
- $X_{max} \simeq X_0 \ln\left(\frac{E_0}{E_{crit}}\right)$
- $X_{max}(A)$, $N_\mu(A)$

gives insight in scaling of particle components

(−) oversimplified (triangle shower)
spectra not considered, all particles same energy

$\boxed{\text{TRIANGLE}}$

Rossi/Greisen



(+)
extremely accurate
tells you that

- $X_{max} \simeq X_0 \ln\left(\frac{E_0}{E_{crit}}\right)$
- $N(X) \simeq 0.31 \sqrt{\frac{X_{max}}{X_0}}^{\,-1} e^{\frac{X}{X_0}\left(1-\frac{3}{2}\ln s\right)}$

⚠ $N_{max} = N_{max}(X_{max})$ unique

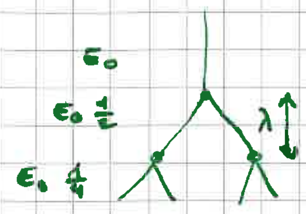
(−) complicated
only $e\gamma$ particles described

$\boxed{\text{ONION}}$

Heitler model (with extension)

"the triangle"

$\varepsilon_c = 87\ \mathrm{MeV}$ $\quad X_0 = 37\ \frac{g}{cm^2}$



equal splitting of the energy

$$\varepsilon_c = \frac{E_0}{2^n} \Rightarrow n = \frac{\ln\left(\frac{E_0}{\varepsilon_c}\right)}{\ln 2}$$

2^n: number of particles

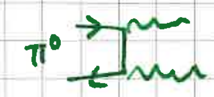
maximum depth is reached at

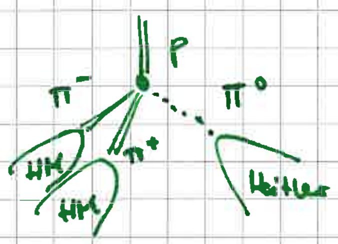
$$N^{max} = \frac{E_0}{\varepsilon_c}$$

$$X_{max} = \underbrace{\ln 2 \cdot \lambda}_{X_0} \cdot \ln \frac{E_0}{\varepsilon_c}$$

what about hadronic ~~different~~ particles?

$$\frac{N_{\pi^0}}{N_{CH}} = \frac{1}{2}$$





$$E_{e\gamma} = \left(1-\left(\tfrac{2}{3}\right)^n\right) E_0 \qquad E_{had} = \left(\tfrac{2}{3}\right)^n E_0$$

$\Rightarrow 2 \cdot N_{\pi^0}$ electromagnetic cascades are initiated with $1/3$ of the energy

$\Rightarrow$ Each cascade from the first interaction carries $\frac{E_0}{3 \cdot N_{CH}}$

$\pi^\pm$ decay to μ (nothing more)

$$X_{max} = (X_1 + \ ) X_0 \ln\left(\frac{E_0}{3 N_{CH}\ \varepsilon_c}\right)$$

$$N_\mu = N_{CH}^{max} = (N_{CH})^{n\ steps} = (N_{CH})^{\left(\frac{\ln\left(\frac{E_0}{\varepsilon_c^\pi}\right)}{\ln\left(\frac{3}{2} N_{CH}\right)}\right)}$$

N_μ grows less than linearly with E_0!

$$= \left(\frac{E_0}{\varepsilon_c^\pi}\right)^{\left(\frac{\ln N_{CH}}{\ln \frac{3}{2} N_{CH}}\right)} = \left(\frac{E_0}{\varepsilon_c^\pi}\right)^{\beta\ (\approx 0.95)}$$

Example:
$E_0 = 10^{18}\ eV$
$\varepsilon_c^\pi = 20\ GeV$
$\varepsilon_c^{e\gamma} = 87\ MeV$
$\beta = 0.95$
$N_\mu^{max} = 20 \cdot 10^6 \approx 0.4 \frac{E_0}{\varepsilon_c^\pi}$
$N_{e\gamma}^{max} = 6.7 \cdot 10^9$

iron is the same as 56 protons!

$$E_0 \rightarrow \frac{E_0}{A}$$

$$N_\mu \rightarrow A\ N_\mu$$

$$X_{max}^{(A)} = X_1 + X_0 \ln\left(\frac{E_0}{3 \cdot A \cdot N_{CH} \cdot \varepsilon_c}\right) = X_{max}^{(1)} - X_0 \ln A$$

$$N_\mu^{(A)} = A \cdot \left(\frac{E_0}{A\ \varepsilon_c^\pi}\right)^\beta = \underbrace{A^{1-\beta}}_{} N_\mu^{(1)}$$

$$\beta_{min} = \frac{\ln 2}{\ln 3} = 0.631$$

$A=56$ ≈ 1.3

$\beta = 0.9 \Rightarrow N_\mu^{max} = 17\% \left(\frac{E_0}{\varepsilon_c^\pi}\right)$

$\beta \rightarrow 1$ (for $N_{CH} \rightarrow \infty$)

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why ~~is this~~ are showers so universal ?

$$N_{e\gamma}^{(A)} = \left(E_0 - \varepsilon_c^{\pi} N_\mu^{max}\right)\frac{1}{\varepsilon_c} = \cancel{\ldots}$$

$$= \left(E_0 - \varepsilon_c^{\pi} A^{1-\beta}\left(\frac{E_0}{\varepsilon_c^{\pi}}\right)^{\beta}\right)\frac{1}{\varepsilon_c}$$

$\frac{\varepsilon_c^{\pi}}{\varepsilon_c}$ is large ! (≈ 1000)

$$\frac{\partial N_{e\gamma}^{(A)}}{\partial A} = \frac{\varepsilon_c^{\pi}}{\varepsilon_c}\left(\underbrace{A^{-\beta}(\beta-1)}_{\approx\, \mathcal{O}(-0.01)}\left(\frac{E_0}{\varepsilon_c^{\pi}}\right)^{\beta}\right)$$

... but still

$$\frac{1}{N_{e\gamma}^{(A)}}\frac{\partial N_{e\gamma}^{(A)}}{\partial A} = A^{-\beta}(1-\beta)\frac{E_\mu}{E_0 - E_\mu}$$

electromagnetic component does not change as a function of A more than $\sim 2\%$ above $E_0 \approx 10^{18}$ eV !

$e\gamma$ is "fueled" just earlier, but with same total energy !

Conclusion :

→ $e\gamma$ component does not change in size as a.f.o. A

→ $e\gamma$ component is shifted (X_{max}) as a.f.o. A

for X_{max} / N_μ

iron shower with E_0 looks the same as 56 proton showers combined with $\frac{1}{56}E_0$, each

$$X_{max} - X_1 = X_0 \ln\left(\frac{E_0}{3\cdot A\cdot N_{ch}\cdot \varepsilon_c}\right)$$

$$\frac{N_\mu}{A} = \left(\frac{E_0}{A\cdot\varepsilon_c^{\pi}}\right)^{\beta}$$

⚠ for the same energy, iron proton differ by:

→ X_{max} is shifted ($\Delta X_{max} = X_0 \ln(A)$)

→ more muons (factor ~ 1.3)

⇒ BUT approx. same number of total particles produced !

π = electron / positron

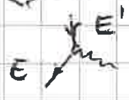
Rossi-Greisen model

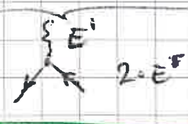
"the onion"

"how many particles of type π and energy E are present at the depth x or what is their rate of change?"

$$\frac{d\pi(E,x)}{dx} = -\frac{\pi(E,x)}{\lambda_{brems}}$$

$$+\underbrace{\int_E^\infty \pi(E',x)\,\frac{d^2 n_{e\to e}}{dx\,dE}\,dE'}_{} + \underbrace{2\int_E^\infty \gamma(E',x)\,\frac{d^2 n_{\gamma\to 2e}}{dx\,dE}\,dE'}_{}$$





APPROXIMATION A!

you can show by substitution that

$$\pi(E,x) = f(E)\cdot g(x) = a\cdot E^{-(s+1)}\,g(x)$$

solves this equation!

$$E^{-(s+1)}\frac{dg(x)}{dx} = g(x)\left[-\frac{E^{-(s+1)}}{\lambda_{brems}} + \int_E^\infty \ldots dE + 2\int_E^\infty \ldots dE'\right]$$

for a fixed energy

$$\frac{1}{g(x)}\frac{dg(x)}{dx} = \lambda(s) \qquad \lambda \text{ is a function of } s \text{ only!}$$

and $g(x) \simeq e^{\lambda(s)\cdot x}$

elementary solution has the form

$$\pi(E,x) = a\cdot E^{-(s+1)}\,e^{\lambda_1(s)\cdot x}$$

$\boxed{\text{see Rossi, Greisen, Gaisser, Engel}}$

from saddle point approx.:

$$\frac{d}{ds}\left(\lambda(s)\,x + s\ln\frac{E_0}{E} + \frac{1}{2}\ln s\right) = 0$$

from spectrum-weighted moments:

$$\lambda \approx \frac{1}{2}\left(s-1-3\ln s\right)$$

$\Longrightarrow$ I: there is a maximum at $s=1$, $\lambda = 0$

⚠ II: $X_{max} = X_0 \ln\frac{E_0}{E}$

all particles above E! put $E = E_c$ for total shower!

III: $s = \frac{3X}{X + 2X_{max}}$

Longitudinal profile:

particles present at depth X with energy $E > E_c$

$$t = \frac{X}{X_0}$$

special solution, one particle incident

Greisen approx.

$$\lambda(s) \approx \frac{1}{2}\left(s - 1 - 3\ln(s)\right)$$

$$X_0 \frac{1}{N(x)} \frac{\partial N(x)}{\partial x} = \frac{\partial \ln(N(t))}{\partial t} = \lambda(s)$$

$$\Rightarrow s = \frac{3t}{t + 2t_{max}} \Rightarrow \frac{\partial \ln(N(t))}{\partial t} = \frac{1}{2}\left(\frac{3t}{t+2t_{max}} - 1 - 3\ln\left(\frac{3t}{t+2t_{max}}\right)\right)$$

integrating ...

$$\ln(N(t)) = n_0 \left(t\left(1 - \frac{3}{2}\ln\left(\frac{3t}{t+2t_{max}}\right)\right)\right)$$

$$\Rightarrow N(t) = N_0 \, e^{t(1-\frac{3}{2}\ln(s))}$$

using boundary condition to fix N_0: $N(t_{Xmax}) \stackrel{!}{=} N_{max} = N_0 e^{t_{max}}$

from calculations under Approx. B (Tamm / Belenky 1946)

$$N_{max}(E_0) = \frac{0.31}{\sqrt{\ln\left(\frac{E_0}{E_c}\right)}} \frac{E_0}{E_c} \Rightarrow t_{Xmax} = \ln\frac{E_0}{E_c}$$

$$N_0 = \frac{0.31}{\sqrt{\ln\left(\frac{E_0}{E_c}\right)}}$$

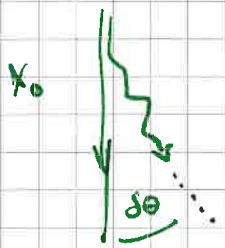
$$N(t) = \frac{0.31}{\sqrt{\frac{X_{max}}{X_0}}} \, e^{t\left(1-\frac{3}{2}\ln(s)\right)}$$

Greisen profile !

repeat using $\lambda(X) \simeq -\frac{1}{\Lambda}\frac{X - X_{max}}{X(-X_1)} = -\frac{1}{\Lambda}\left(1 - \frac{X_{max}}{X}\right)$

for Gaisser-Hillas (but there $N_{max} \neq N_{max}(X_{max})$)

Lateral development of the shower



$$\langle \delta\theta^2 \rangle = \left(\frac{E_s}{E}\right)^2 dt \qquad E_s = \frac{m_e c^2}{\sqrt{\frac{\alpha}{4\pi}}} \simeq 21\,\mathrm{MeV}$$

$$r_M = \frac{E_s}{E_{crit}} X_0 \simeq 10 \frac{g}{cm^2}$$

Solve cascade equations for $\pi(E, X, r, \vartheta)$ $\longrightarrow$ many tried, two succeeded! ~~only two people have ever done this!~~

Nishimura, Kamata ~ 1950s

approximation:

$$\frac{r}{r_M} f\left(\frac{r}{r_M}\right) \sim \left(\frac{r}{r_M}\right)^{s-1} \left(1 + \frac{r}{r_M}\right)^{s-\frac{7}{2}}$$ normalized such that

$$1 \stackrel{!}{=} 2\pi \int_0^\infty \left(\frac{r}{r_M}\right) f\left(\frac{r}{r_M}\right) d\left(\frac{r}{r_M}\right)$$

NKG function $\Rightarrow$

$$f(t, r) = \frac{N(t)}{2\pi r_M^2} \frac{\Gamma(\frac{9}{2} - s)}{\Gamma(s)\,\Gamma(\frac{9}{2} - s)} \left(\frac{r}{r_M}\right)^{s-2} \left(1 + \frac{r}{r_M}\right)^{s-\frac{9}{2}}$$

which for pure electromagnetic showers

given a function $N(t)$, the NKG actually describes the whole shower!

however: $$\lambda(s) \cdot t + \ln\left(\frac{E_0}{E_c}\right) + \underbrace{\ln\left(\frac{r}{r_M}\right)}_{\text{new term!}} \stackrel{!}{=} 0$$

$$s = \frac{3t}{t + \underbrace{2\ln\frac{E_0}{E_c} + 2\ln\left(\frac{r}{r_M}\right)}_{\text{retarded } X_{max}\text{!}}}$$

a universal model of particle densities

given the fact that particles in air showers

- scale in terms of number $\sim \left(\frac{E_0}{E_{ref}}\right)^{\gamma}$ (Heitler – Matthews)
- produce universal long. profiles $\frac{1}{x_0}\lambda(s) \sim \frac{1}{g(x)}\frac{\partial g}{\partial x}$ (Rossi – Greisen)
- follow the same lateral distr. $\sim (r)^{s-2}(1+r)^{s-\frac{9}{2}}$ (Nishimura Kamata Greisen)
- have universal ~~energy~~ spectra $d\sigma/dE \sim E^{-(s+1)}$ (Rossi, Engel)
- all of the above is fulfilled not just on average but for almost every shower ! (Hillas)

we can write down a model of particle densities as

$$\rho \simeq \left(\frac{E_0}{E_{ref}}\right)^{\gamma} g(\Delta x)\, f(r)$$

$$\Delta x = X - X_{max}$$

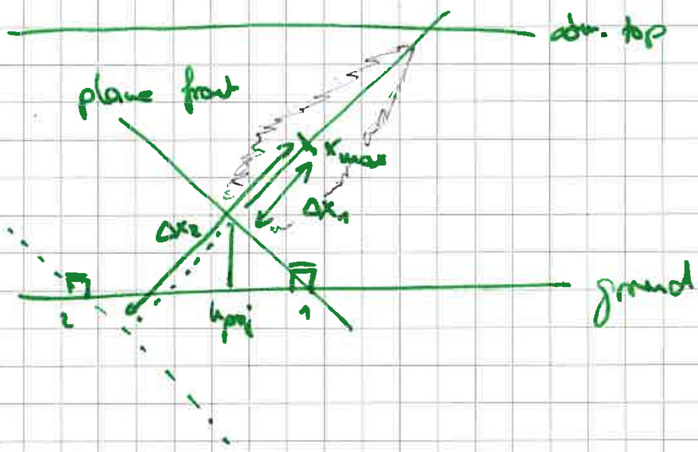
⚠ all of the above is not valid for $\Delta x \ll 0$, but for $\Delta x \gtrsim 0$

ρ describes the density of particles in the shower plane at the distance Δx from the maximum, with distance r from the shower axis

the depth parameter ΔX

shower development in terms of universality is expressed relatively to X_{max}

$\Rightarrow \quad \Delta X = X - X_{max}$



$$\Delta X \approx \frac{X_{vg}}{\cos\theta}\, e^{-\frac{h_{proj}}{h_s}} - X_{max}$$

$h_{proj} = h + r \sin\theta \cos\psi$
height of the shower core!

X_{vg}, h_s seasonally dep.,
$h_s \approx h_s^{(0)} + h_s^{(1)} \cdot h$

Gaisser-Hillas in ΔX:

$$N(\Delta X) = N_{max} \left(\frac{\Delta X}{X_{max}} + 1 \right)^{\frac{X_{max}}{\lambda}} e^{-\frac{\Delta X}{\lambda}}$$

(for $X_1 = 0$)
$X_{max} \rightarrow X_{max} - X_1$

full model!

$$\rho = \left(\frac{E_0}{E_{ref}} \right)^{\gamma} g(\Delta X) \cdot f(r)$$

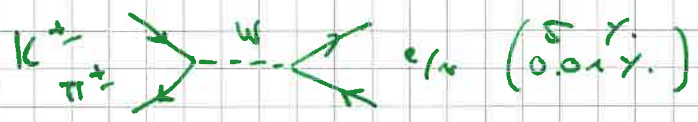
particle density in plane at ΔX at distance r

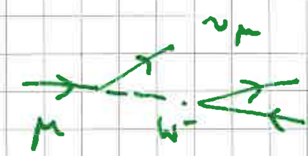
GH-like

NKG-like

four components!

$\Rightarrow$ many of the previous calculations have been done before the discovery of the pion (1947) and way before all particle decay channels were understood





^ after $\gamma \cdot 600\,\mathrm{m}$

$$\gamma = \frac{E}{100\,\mathrm{MeV}}$$

$$\gamma \approx 10 \dots 1000 \dots$$

$$E^{tot}_{had/\mu} = \varepsilon_c^\pi \left(\frac{E_0}{\varepsilon_c^\pi} \right)^\beta$$

assume a fraction m "leaks" from the hadronic / muon component back into the em particles!

$$\Delta E_{e\gamma} = m \cdot E^{tot}_{had/\mu} \Rightarrow N^{tot}_{e\gamma} = \Big(E_0 - \underbrace{\varepsilon_c^\pi N_\mu^{max}}_{\approx E^{tot}_{had/\mu}} + m\, E^{tot}_{had} \Big) \frac{1}{\varepsilon_c}$$

$$N^{tot}_{e\gamma} = \left(E_0 - (1-m)\, E^{tot}_{had/\mu} \right) \frac{1}{\varepsilon_c}$$

$$\Delta N_{e\gamma} = m \cdot \frac{\varepsilon_c^\pi}{\varepsilon_c} \left(\frac{E_0}{\varepsilon_c^\pi} \right)^\beta$$

at $E_0 \approx 10^{17}$ eV $\quad \dfrac{\Delta N_{e\gamma}}{N_{e\gamma}^{tot}} \approx m$

$\Rightarrow$ up to 10% particles more, but with <u>different</u> long. / lat. profiles and spectra!

$\Rightarrow$ need for disentanglement!

how to parametrize ?

muons (and their decay products) do not look like what we expect from eγ particles in terms of their longitudinal / lateral profiles !

$\Longrightarrow$ modified GH function

$$g(\Delta x) = f_{ref}(r) \left(\frac{\Delta x - \Delta x_1}{\Delta x_{ref} - \Delta x_1} \right)^{\frac{\Delta x_{max} - \Delta x_1}{\lambda}} e^{-\frac{\Delta x - \Delta x_{ref}}{\lambda}}$$

two important features:

1. $g(\Delta x_{ref}) = f_{ref}(r)$
2. $g'(\Delta x_{max}) = 0$

scale / normalization and position of maximum are disentangled !

with this one can parametrize the retardation of the shower maximum w.r.t. distance r from shower axis

$$\Delta x_{max} \approx \Delta x_{max}^{(0)} + \underbrace{r \cdot \Delta x_{max}^{(1)}}$$
$$\lambda \approx \lambda^{(0)} + \underbrace{r \cdot \lambda^{(1)}}_{\text{only important for } e\gamma(\mu)/\mu \text{ components}}$$

this equation exists in slightly modified forms (e.g. R_μ etc.)

from Heitler we assume
$N_\mu \sim N_{Had} \sim N_{e\gamma/\mu/Had}$

$$R_\mu = \frac{N_\mu}{\langle N_\mu^p \rangle}$$

at $\Delta x = \Delta x_{ref}$

$\Longrightarrow$ $f_i = (a_i (R_\mu - 1) + 1)$ for each !

$$\boxed{f_{tot} = \sum_i (a_i (R_\mu - 1) + 1)}$$

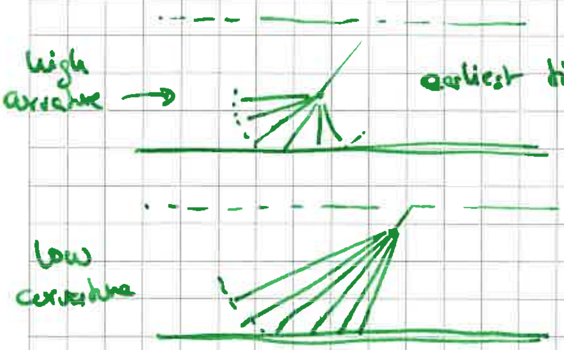
This describes not only different primaries now, but also approx.ly different hadronic interaction models !

~~fo~~ model of particle densities ("signal model") works nice to describe expected det. responses as a function of R_μ and X_{max}. assume $\rho \sim S$!

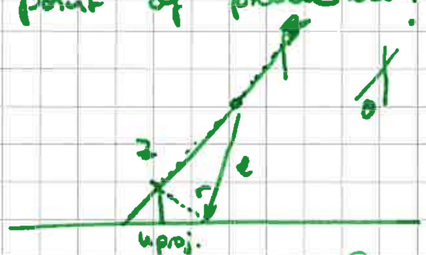
$$R_\mu , X_{max} \longmapsto S(R_\mu , X_{max})$$

but inverse is tricky! What is $R_\mu(S)$ / $X_{max}(S)$?

⇒ muons propagate (almost) rectilinearly!



earliest time (horizon) given by sphere expanding with c for a single point of production!



When ~~do~~ do particles arrive at a detector?

$\ell = \sqrt{z^2 + r^2}$ express time relative to plane front!

plane front travels a distance of z with speed c, so particle at time t fulfills

$$ct = \ell - z \Longrightarrow z = \frac{1}{2}\left(\frac{r^2}{ct} - ct\right)$$

$h_{proj} = r \sin\theta \cos\psi$

thus the particle was produced at the height

$$h(t) = z\cos\Theta + h_{proj} = \frac{1}{2}\left(\frac{r^2}{ct} + ct\right)\cos\theta + h_{proj}$$

assuming isothermal atmosphere

$$X(t) = \frac{X_{vg}}{\cos\Theta} e^{-\frac{h(t)}{h_s}} = \frac{X_{vg}}{\cos\theta} \exp\left[-\frac{1}{h_s}\left(\frac{1}{2}\left(\frac{r^2}{ct} + ct\right)\cos\theta + h_{proj}\right)\right]$$

We assume particles are being produced also according to GH profile!

More than one particle is of course created! We want to calculate

$$\frac{d^2 n}{dt\, dr} = |J| \frac{dN(x(t))}{d \sin\alpha}$$

not all ~~particles~~ angles equally covered

$$\frac{d^2 N}{d\sin\alpha\, dE} = E \frac{d^2 N}{dE\, dp_T} \qquad p_T \simeq E \sin\alpha$$

$$= (\ldots)\; E^{-(s+1)}\, e^{-\frac{p_T}{Q}}$$

$$\frac{dN}{d\sin\alpha} = (..)\left(\frac{m}{Q}\right)^s I(\sin\alpha) \qquad \sin\alpha = \frac{r}{e}$$

Jacobean from coordinate transformation:

$$|J| = \left| \frac{dx(t)}{dt}\frac{d\sin\alpha}{dr} - \frac{dx(t)}{dr}\frac{d\sin\alpha}{dt} \right|$$

$$= (\ldots) = \frac{x(t)}{h_s t}\left(\frac{r^2 - (ct)^2}{r^2 + (ct)^2}\right)^2$$

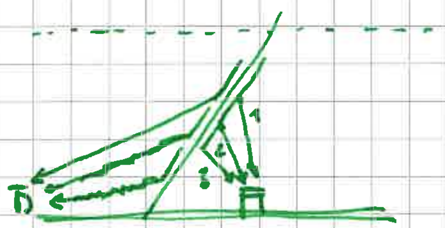
$$\frac{d^2 n}{dt\, dr} = N_{max}\, s \left(\frac{m}{Q}\right)^s I(r,t)\, X_{vg} \frac{(r^2 - (ct)^2)^2}{h_s t\, (r^2 + (ct)^2)^2} \left(\frac{2ctr}{r^2 + (ct)^2}\right)^{s-1}$$
$$e^{\left(-\frac{h(t)}{h_s} + \frac{X_{max}}{\lambda} - \frac{X_{vg}}{\lambda\cos\theta} e^{-\frac{h(t)}{h_s}}\right)} \left(\frac{X_{vg}}{X_{max}\cos\theta} e^{-\frac{h(t)}{h_s}}\right)^{\frac{X_{max}}{\lambda}}$$

this can be (needs to be) approximated by a log-normal distribution function!

for the "time mode" we care about where particles are produced rather than where they are / how many are present! for muons this works nicely, for e,γ not so much.

but here is a trick! where are most particles produced? ⇒ X_{max}!

furthermore, assume time ordering!



$t_1 < t_2 < t_3$

even with energy loss and non-rectilinear movement, time ordering should be approximately preserved!

this means the integral is preserved!

$$\int_0^{t_q} n'(t)\, dt \sim \int_0^{X_q} g'(x)\, dX$$

so when do particles produced at X_{max} arrive?

$$\frac{1}{(\ldots)} \int_0^{X_{max}} g(x)\, dx \simeq \underline{\underline{40\,\%}}$$

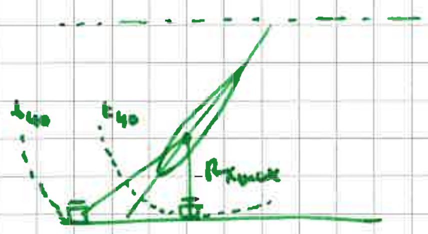
$$\left(= 1 - \frac{\frac{\lambda}{X_{max}}\, \Gamma\left(\frac{X_{max}+\lambda}{\lambda}, \frac{X_{max}}{\lambda}\right)}{\Gamma\left(\frac{X_{max}}{\lambda}\right)} \right)$$

following the rational from muon production depth (MPD) procedure, t_{40} (relative to plane front) is the important ~~time~~ point in time for X_{max}!

assume isothermal atmosphere,

$$X(t_{40}) \approx X_{max} = \frac{X_{rg}}{\cos\Theta} \exp\left[-\frac{1}{2}\left(\frac{r^2}{ct_{40}} - ct_{40}\right)\frac{\cos\Theta}{h_s}\right] e^{-\frac{h_{proj}}{h_s}}$$

remember ΔX ! $$= (\Delta X + X_{max}) \exp\left[-\frac{1}{2}\left(\frac{r^2}{ct_{40}} - ct_{40}\right)\frac{\cos\Theta}{h_s}\right]$$



$$ct_{40} = \sqrt{R_{X_{max}}^2 + r^2} - R_{X_{max}}$$

$$R_{X_{max}} = -\frac{1}{2}\left(\frac{r^2}{ct_{40}} - ct_{40}\right) = \frac{h_s}{\cos\Theta}\ln\left(\frac{\Delta X}{X_{max}} + 1\right)$$

$$\left(R_{X_1} = \frac{h_s}{\cos\Theta}\ln\left(\frac{\Delta X + X_{max}}{X_1}\right)\right)$$

log-normal-dist. function expressed in terms of t_{40}:

$$\frac{dn}{dt} = \frac{n_{tot}}{\sqrt{2\pi}\,\sigma\,(t-t_0)}\, e^{\left(-\frac{1}{2\sigma^2}\left(\ln\left(\frac{t-t_0}{t_{40}-t_0}\right) + \sqrt{2}\,\sigma\, \mathrm{erf}^{-1}(2\cdot 0.4 - 1)\right)^2\right)}$$

this is an idealized function, which must be treated with correction terms and then parametrized!

use n_{tot} from areal density model, then

$$dS(t) = \sum_{i,\,\text{components}} a \frac{dn_i}{dt}$$

that's it !