Quarks, Partons, and QCD: Lecture 12

Mass singularities in QCD - Jets

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KLN theorem

Kinoshita-Lee-Naunberg

Consider the general scattering process $A \to B$ between states A and B of massive particles. It may happen that the scattering matrix $|S_{AB}|$ has mass singularities but if we sum the squares of scattering amplitudes

$$\sum_{D(A),D(B)} |S_{D(A),D(B)}|^2$$

over the sets $D(A)$, $D(B)$ of states degenerate with A and B, the sum has no mass singularities, i.e. is finite even for massless particles.

- **degenerate states** - all states of particles that have the same quantum numbers as well as the same total four-momentum — experimentally cannot be distinguished

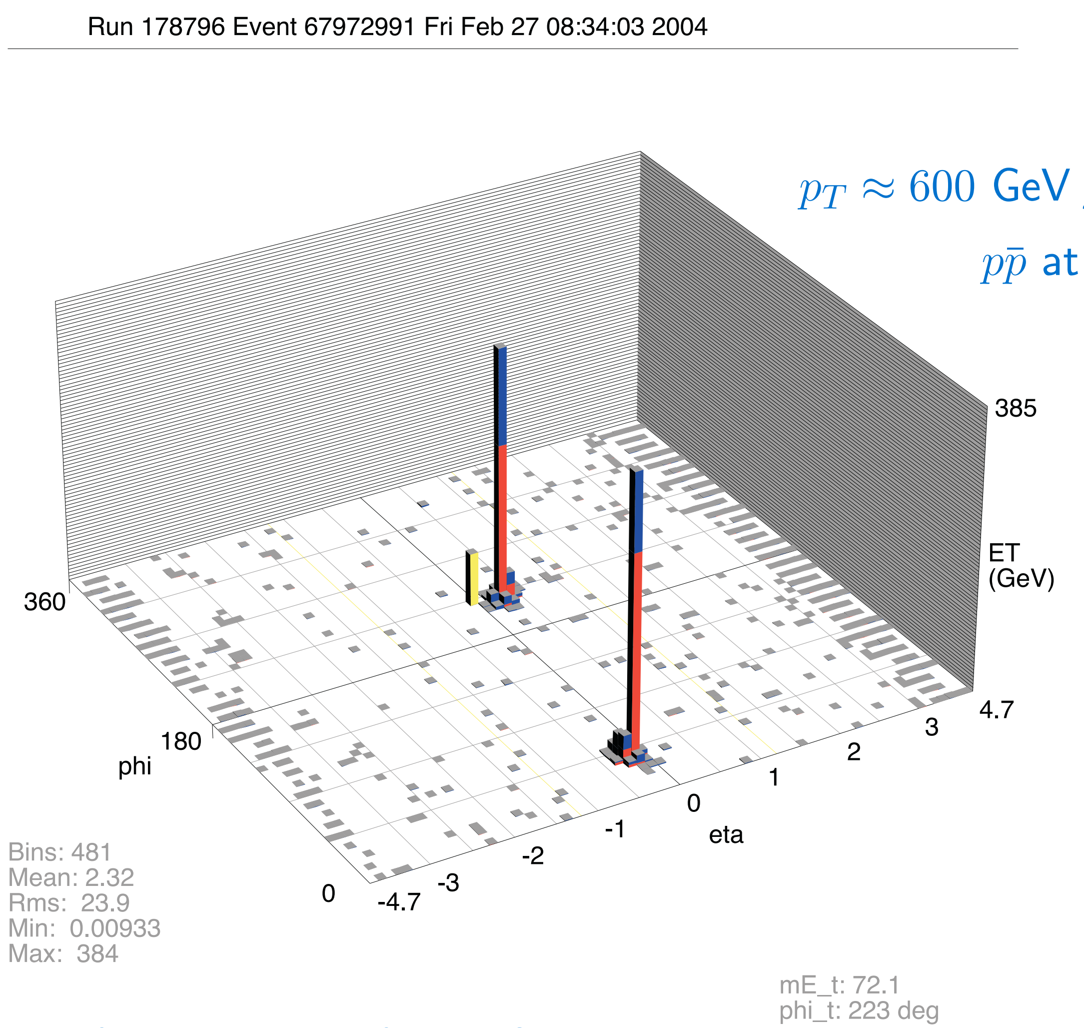
- as electron and quarks have masses, only γ/g mass singularities are present in QED/QCD
- however $m_q \ll \Lambda_{\text{QCD}} \ll Q_{\text{process}}$ + confinement + experimental resolution is usually worse than m_q
- the KLN theorem and techniques how to deal with mass singularities very useful in QCD
- we cannot characterize the final state by number of partons

Jets

- to get more detailed description of final state

  (besides some global variables as event shapes)

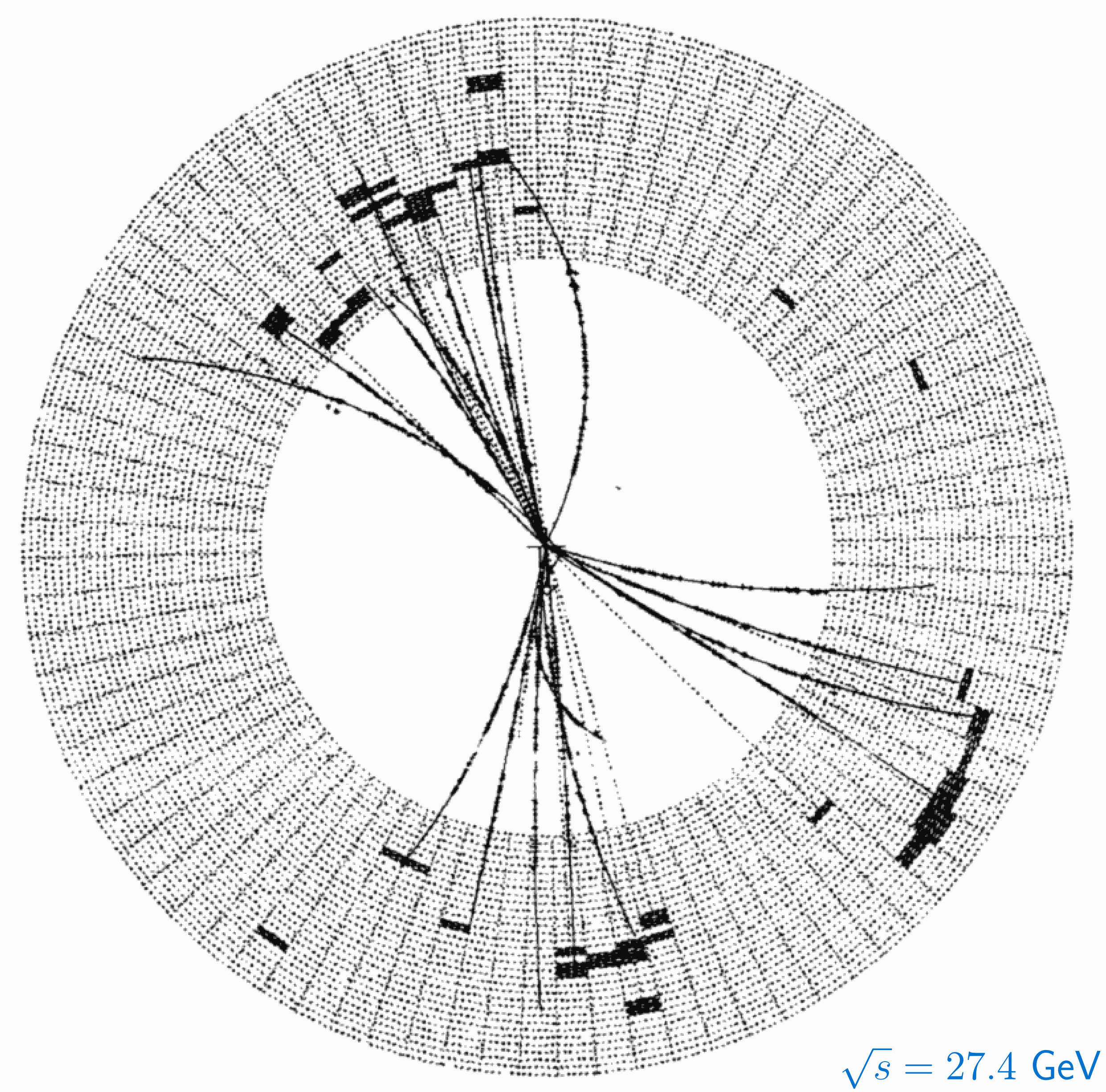
  we need to introduce new concepts that put all degenerate states into one object
- experimentally, hard kicked partons manifest themselves as collimated sprays of particles - **jets**



$p_T \approx 600$ GeV jets at Tevatron

$p\bar{p}$ at $\sqrt{s} = 1.96$ TeV

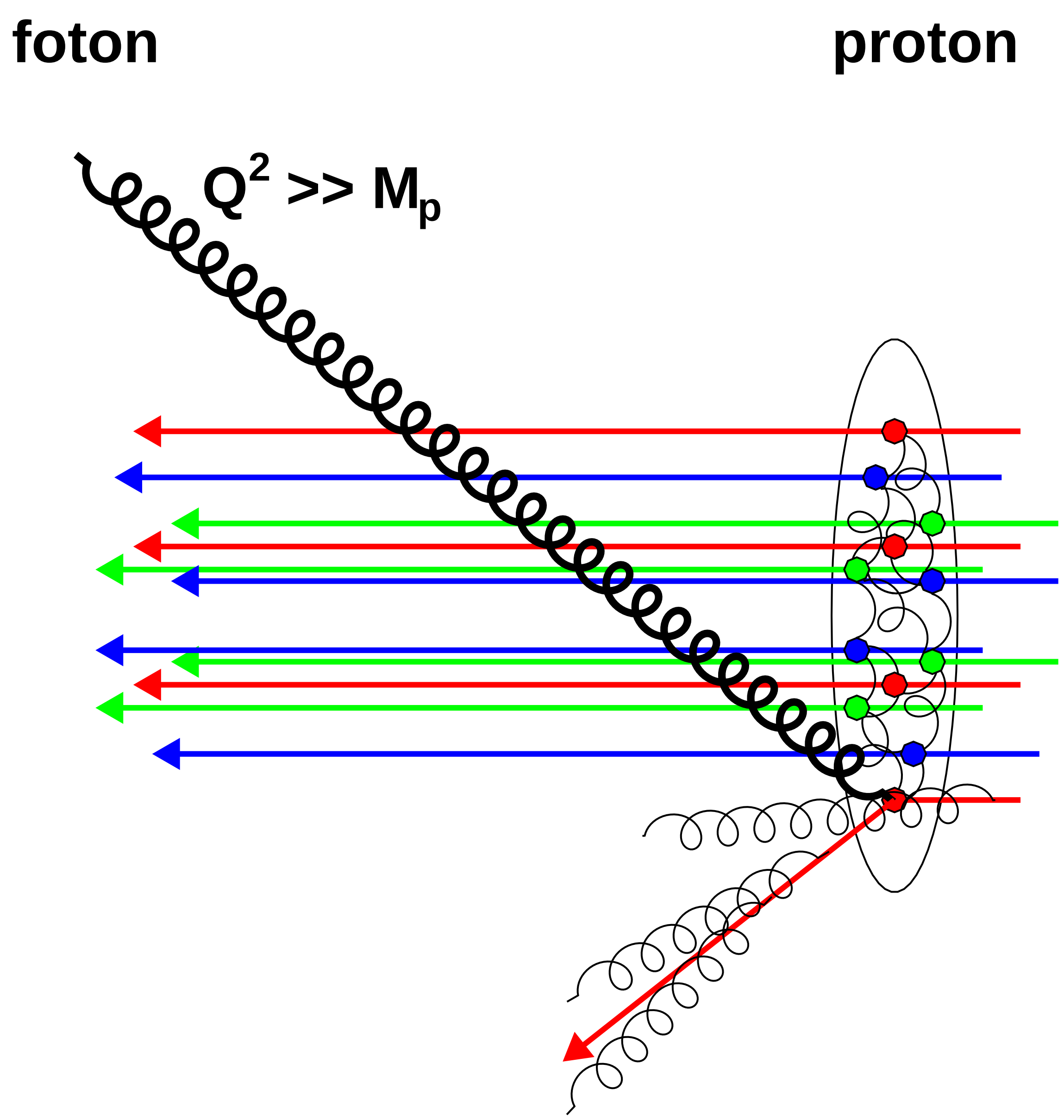
Gluon discovery in e^+e^-



$\sqrt{s} = 27.4$ GeV

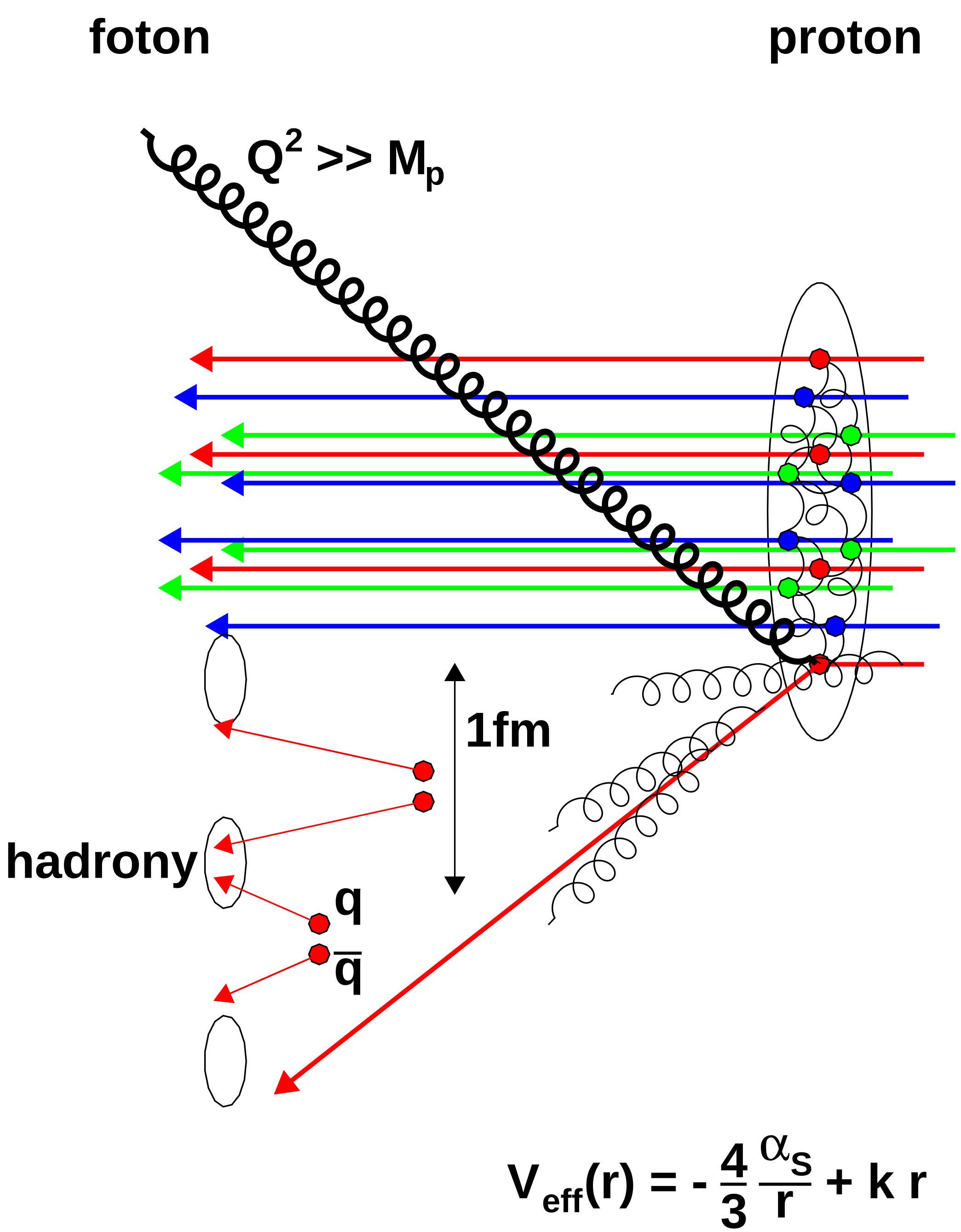
Fig. 11.12 A three-jet event observed by the JADE detector at PETRA.

Hadronization - early stage (parton shower)



- What is happening to the kicked parton?
  - abrupt change of state of motion of color charge
  - in classical picture this implies large acceleration
  - as in QED, accelerated strong force charge (color) radiates
  - as in QED, the radiation is **strongly collimated** in the direction of motion
  - at this early stage, the kicked parton is almost free

  $\Rightarrow$ we can apply pQCD - **parton showers**



Hadron shower

- at the distance of ~ 1 fm (corresponding to $\Lambda_{\mathrm{QCD}} \sim 200$ MeV), the color charges start to feel themselves strongly

$\Rightarrow$ creation of color-anticolor pair from vacuum is energetically more favorable

- process repeats until only colorless objects (hadrons) are formed
- we observe collimated spray of hadrons - **jet**

Jet algorithm

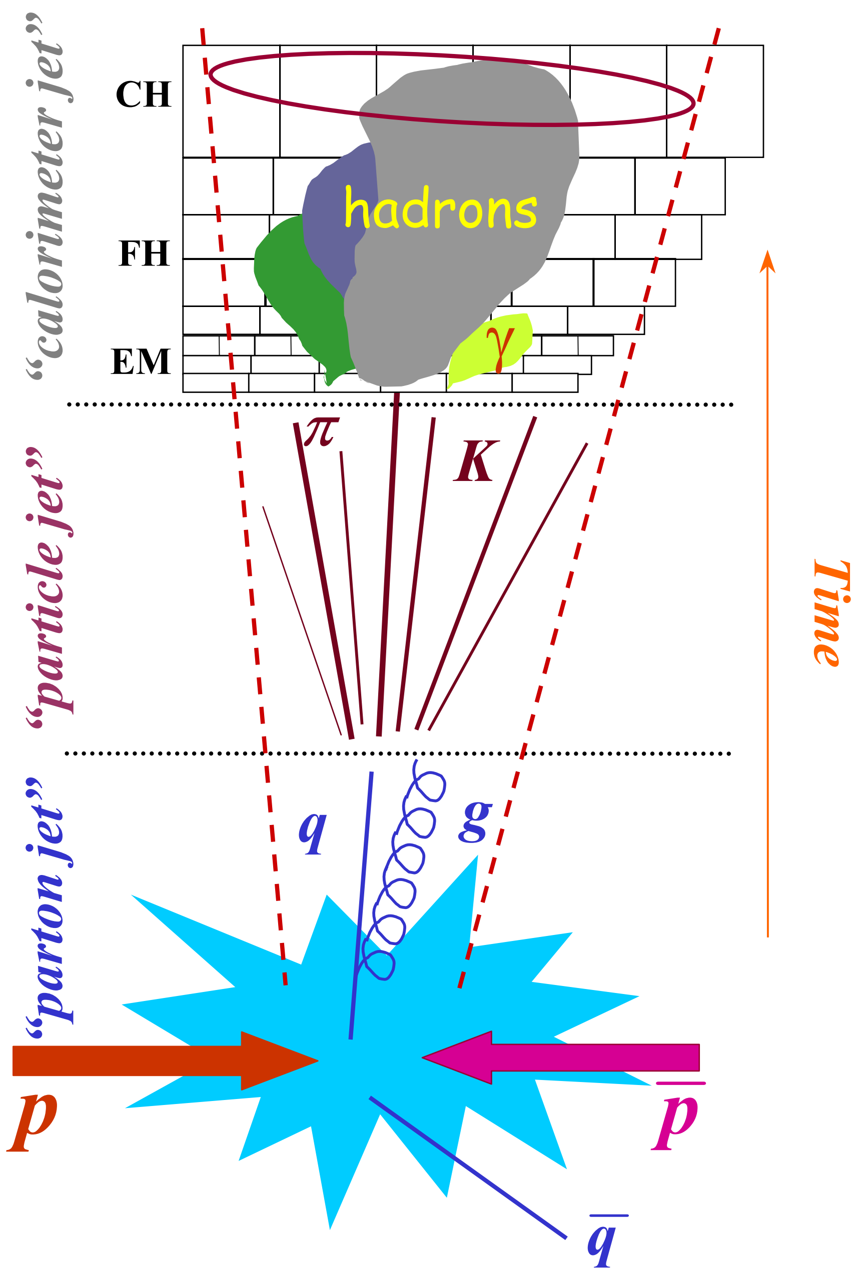
- defines objects that form a jet (jet constituents)

  calorimeter; tracks; hadrons; partons (fixed order pQCD, after parton shower)
- defines how to construct jet 4-momentum from its constituents
- never exact correspondence between kicked parton and jet
- parton is color connected with the rest of proton

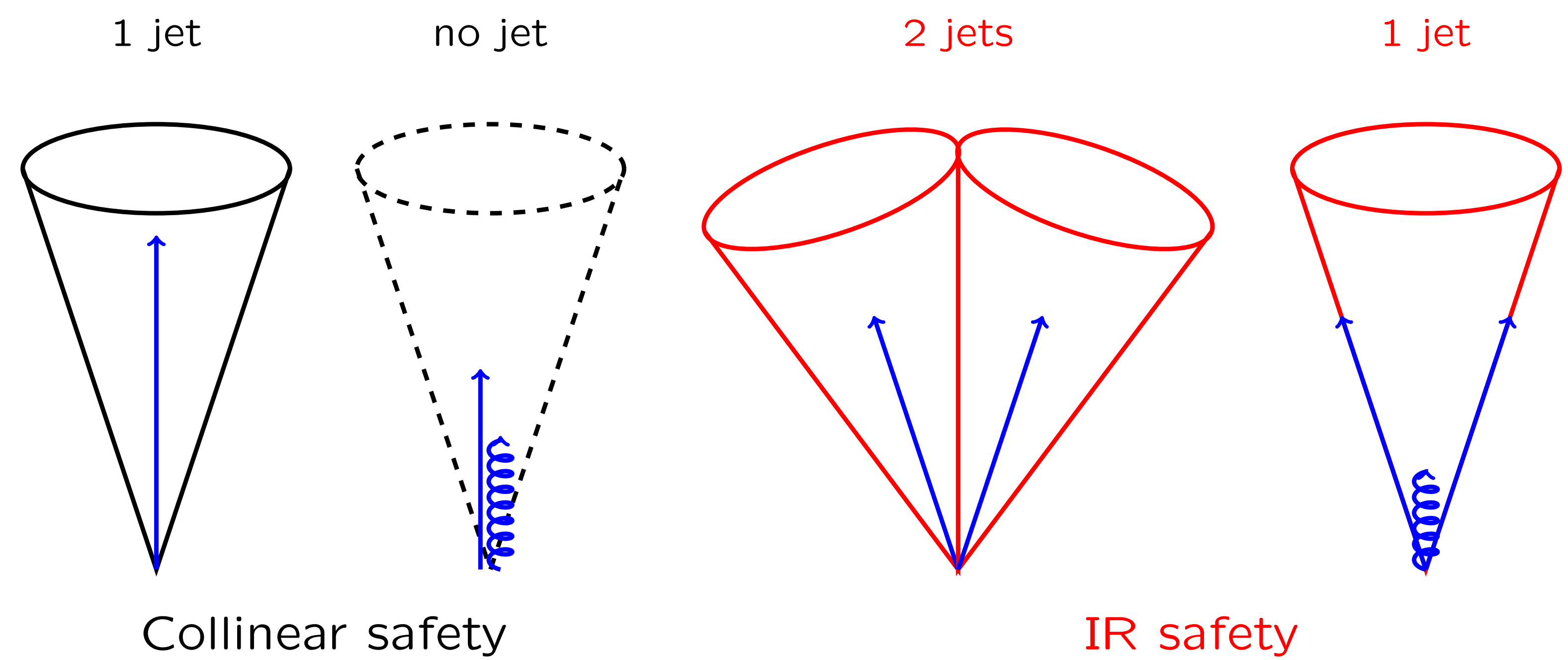
  $\Rightarrow$ cannot assign unambiguously hadrons to partons

  (problem with meassuring m_{top})

Requirements on jet algorithms



- identical procedure on parton, hadron and detector levels
- suitable theoretically (not sensitive to mass singularities)
  collinear splitting and soft emissions should not change jets



- minimal sensitivity to hadronization, underlying event, pile-up
  we have only phenomenological models to correct for these effects
- suitable experimentally
  computing complexity, calibration and correction

JADE algorithm

- example of **sequential clustering algorithm**
- first define *distance* d_{ij}

  for JADE: $d_{ij} = m_{ij}^2/s = y_{ij} = \frac{2E_iE_j}{s}(1-\cos\vartheta_{ij})$
- introduce parameter that controls jet size y_{cut}

(1) find the closest pair $d_{AB} = \min_{i,j} d_{ij}$

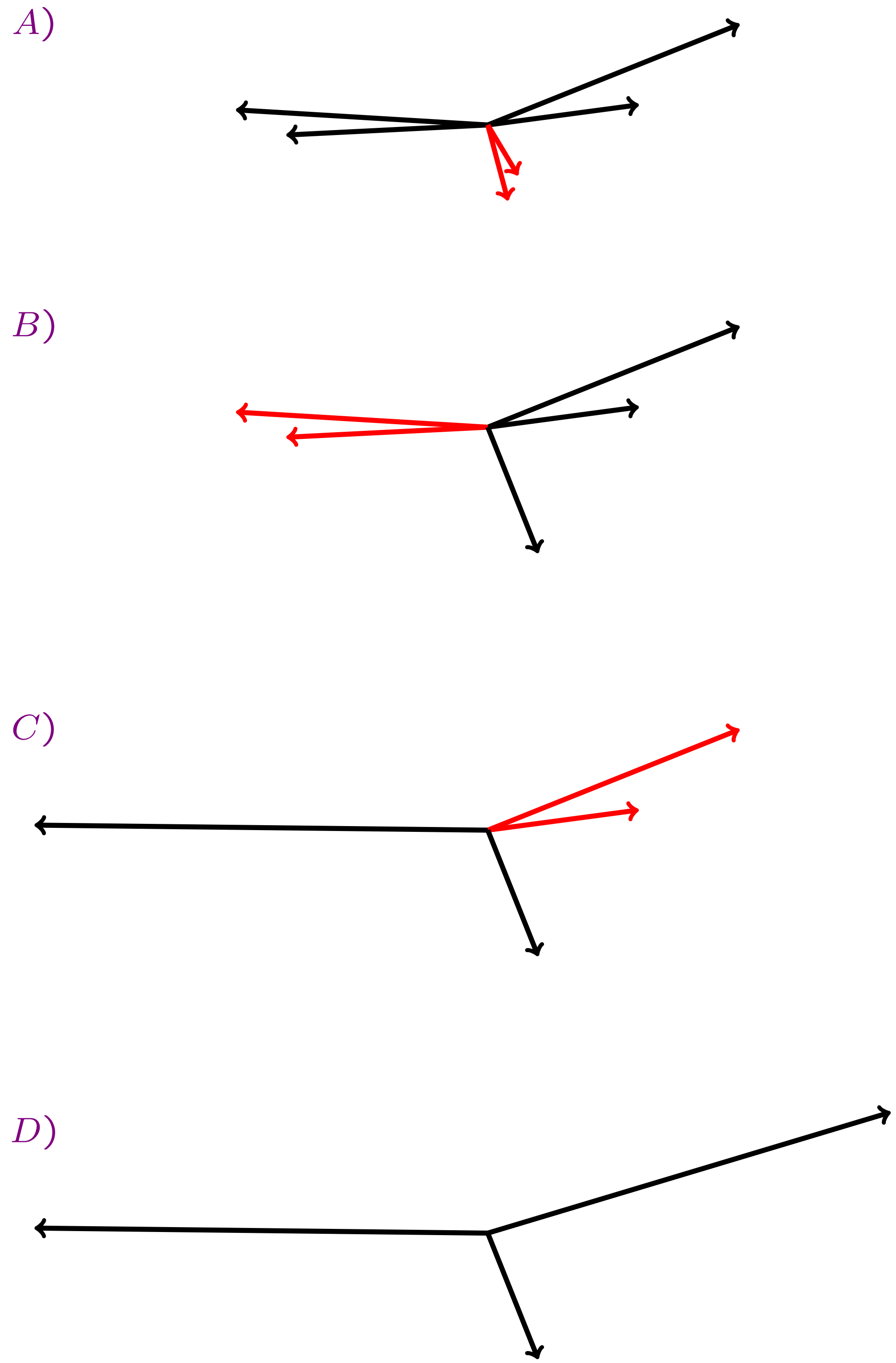
- if $d_{AB} < y_{\text{cut}}$, combine A and B into one particle

$$p_{AB} = p_A + p_B$$

  and repeat the step from the beginning (1)

- if $d_{AB} > y_{\text{cut}}$, algorithm ends (step 2)

(2) particles that are left are called jets



k_T algorithm

$$d_{ij} = \frac{2\min(E_i^2, E_j^2)}{s}(1-\cos\vartheta_{ij}) \approx \frac{\min(k_{Ti}^2, k_{Tj}^2)}{s}$$

- more natural clustering of soft particles
- widely used in e^+e^-

Three jet cross section in e^+e^- for JADE algorithm (ex 7.3)

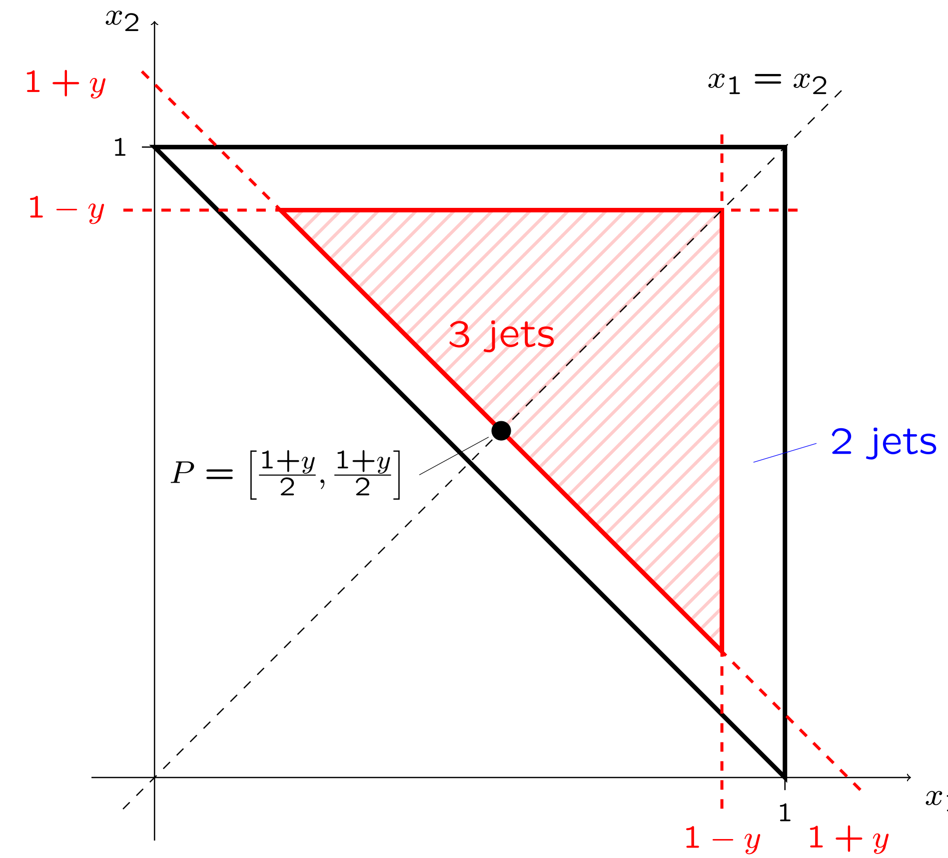
- first contribution from $ee \to q\bar{q}g$

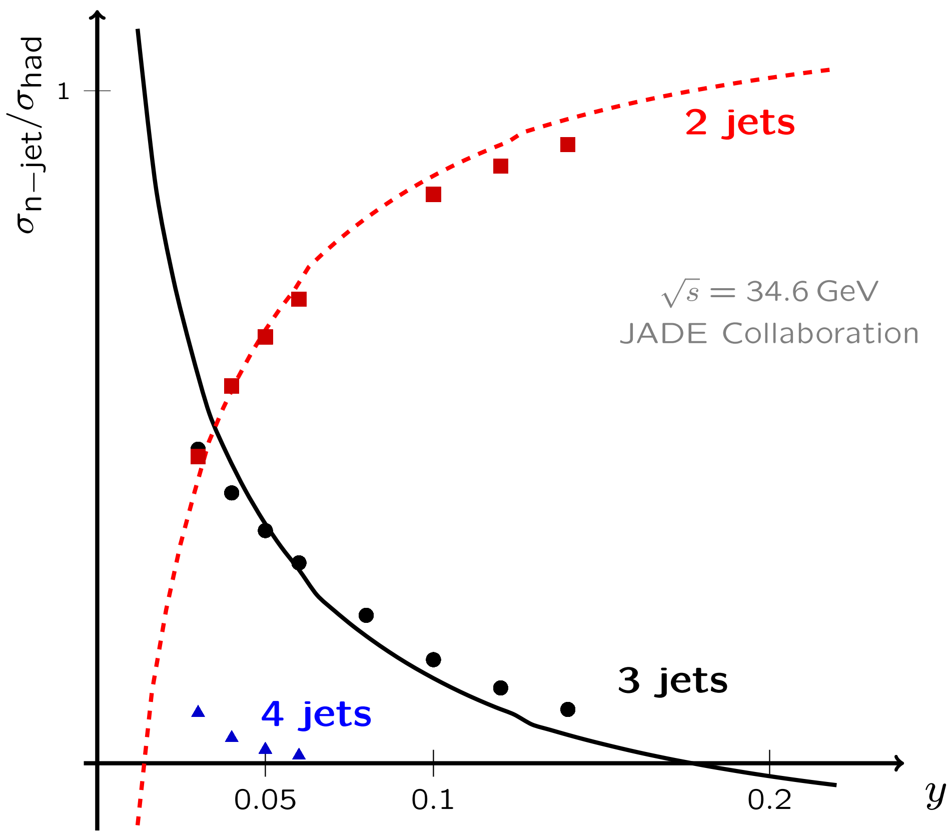
$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- 3-jet phase space $y_{ij} = (1 - x_k) > y_{\text{cut}} = y$

$$\sigma_{\text{3jet}} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \int_{\text{3jet}} dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$\sigma_{\text{3jet}} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \left(2\ln^2 y + 3\ln y + \frac{5}{2} - \frac{\pi^2}{3} + \mathcal{O}(y) \right)$$





$$\sigma_{\text{had}} = \sigma_{\text{2jet}} + \sigma_{\text{3jet}} = \sigma_0(1 + \alpha_s/\pi)$$

$$\Rightarrow \sigma_{\text{2jet}} = \sigma_0 + \sigma_0 \frac{\alpha_S}{2\pi} C_F \left(-2\ln^2 y - 3\ln y - 1 + \frac{\pi^2}{3} + \mathcal{O}(y) \right)$$

- KLN theorem at work
  - for well defined final state configurations (putting degenerate states together) we obtain finite (albeit sometimes negative!) x-sections

Three jet cross section in collinear approximation (ex. 8.3)

Let's first consider a radiation of gluon from the antiquark ($x_1 \to 1$). Radiation is described by

$$\mathrm{d}P = \frac{\alpha_S}{2\pi}\frac{\mathrm{d}t}{t}P_{qq}(x)\mathrm{d}x.$$

As $y_{23} = m_{23}^2/s = t/s$, we have $\mathrm{d}t/t = \mathrm{d}y_{23}/y_{23}$, and gluon forms third jet if $y_{23} > y$.

We still need the integration region for $x_2 = x$. This comes from condition $x_1 > x_2$ (collinear approximation when gluon is radiated from antiquark). However, once we fixed the invariant mass y_{23}, we fixed x_1 as well as the two quantities are related through $y_{23} = 1 - x_1$. This implies that $x_2 < 1 - y_{23}$.

Taking into account radiation from quark (factor of 2),

$$\sigma_{\text{3jet}}(y) = 2\sigma_0\frac{\alpha_S}{2\pi}C_F\int_y^1 \mathrm{d}y_{23}\frac{1}{y_{23}}\int_0^{1-y_{23}}\mathrm{d}x\frac{1+x^2}{1-x}.$$

This leads to,

$$\sigma_{\text{3jet}}(y) = 2\sigma_0\frac{\alpha_S}{2\pi}C_F\left[\ln^2 y + \frac{3}{2}\ln y + 2 + \mathcal{O}(y)\right] = \sigma_0\frac{\alpha_S}{2\pi}C_F\left[2\ln^2 y + 3\ln y + 4 + \mathcal{O}(y)\right].$$

For fun, try to estimate 4-jet cross section in collinear approximation.

Sequential clustering algorithms for hadron colliders

- in addition to a distance between particles i and j

$$d_{ij} = \min\left(p_{Ti}^{2p}, p_{Tj}^{2p}\right)\frac{\Delta_{ij}}{D}, \qquad \text{with} \quad \Delta_{ij}^2 = (y_i - y_j)^2 + (\varphi_i - \varphi_j)^2$$

- introduce a distance between particle and beam $d_{iB} = p_{Ti}^{2p}$
- note that the distances are invariant with respect to the boost along the beam axis

  longitudinal momenta of colliding partons are not known

Clustering Algorithm for hadron colliders

(1) compute all the distances d_{ij} and d_{iB} and find the smallest one
  - if the smallest is d_{ij}, combine particles i and j into one and start from the beginning
  - if the smallest is d_{iB}, remove particle from the list and call it a **jet**

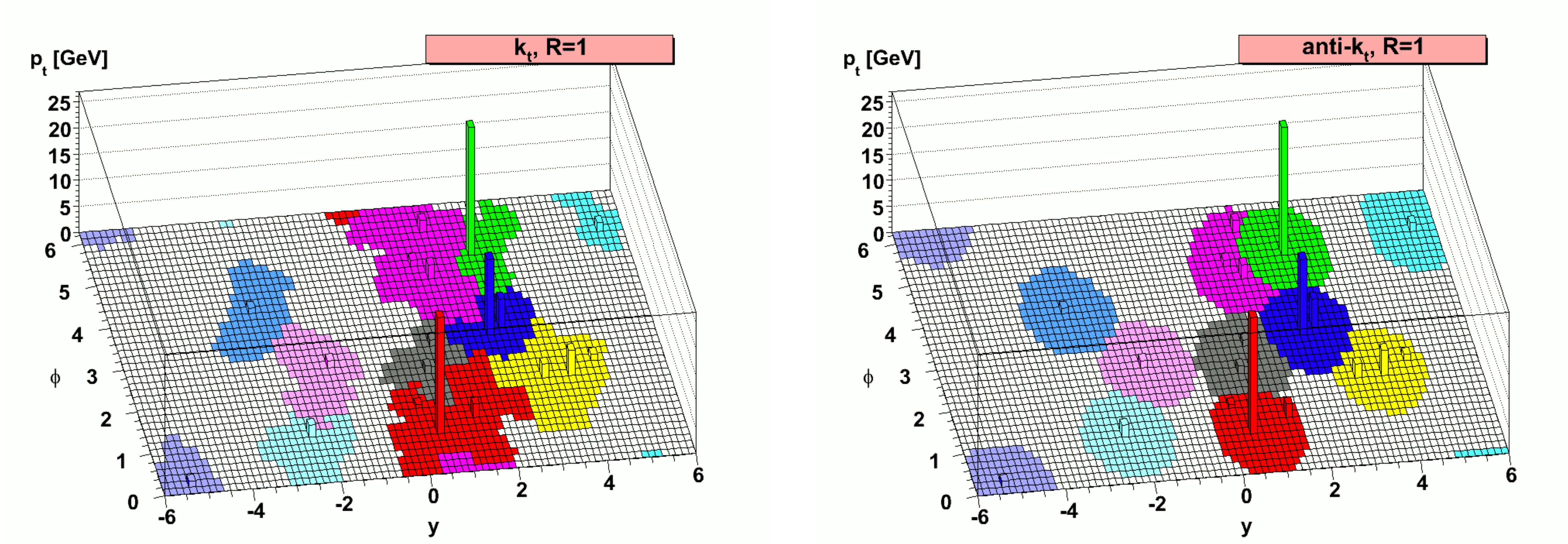
(2) repeat step (1) until all particles are clustered into jets

- parameter D provides a scale between d_{ij} and d_{iB} and controls the size of jets

  distance between jets is always larger than D, $\Delta_{ab} > D$
- parameter p controls the energy power in the distance

  $p = 1$: k_T, $p = 0$: Cambridge/Aachen, $p = -1$: **anti-k_T** (M. Cacciari, G. Salam, G. Soyez: JHEP 0804:063,2008)

k_T versus anti-k_T at hadron colliders



JHEP 0804:063,2008

- hadron colliders - busy environment

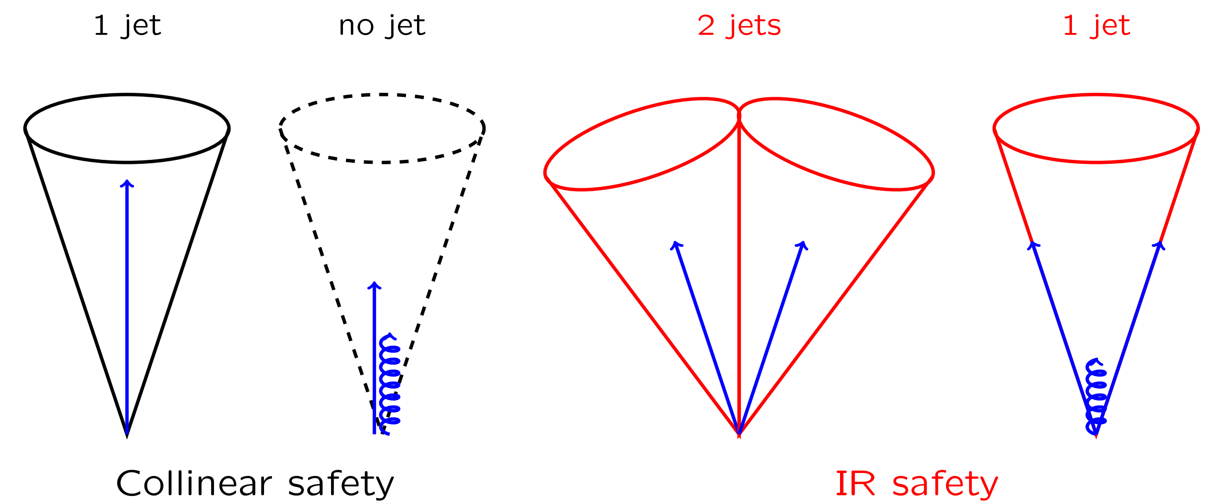
  large pileup, on average 36 pp interactions in one bunch crossing at LHC in 2018
- k_T algorithm has tendency to cluster soft particles from additional interactions into jets, producing large, strange-shaped jets
- slow implementation at Tevatron (greatly improved in FASTJET package)
- **cone algorithms** based on fixed cone shape were used traditionally at hadron colliders (Tevatron)

  they have their own problems (discussed on next slide)
- **anti-k_T** solved the problem - it is now main jet algorithm used by LHC experiments

Cone algorithms

- draw a cone of radius R in distance plane $\Delta_{ij}^2 = (y_i - y_j)^2 + (\varphi_i - \varphi_j)^2$, or $\Delta_{ij}^2 = (\eta_i - \eta_j)^2 + (\varphi_i - \varphi_j)^2$

Iterative Cone Algorithm



(1) find the particle with highest p_T (seed)

(2) draw cone of radius R around the seed (trail jet)

(3) compute new jet axis ($\vec{p}_{\text{jet}}$) from trial jet constituents

(4) if jet axis moves (within precision), draw new cone of radius R around the jets axis and repeat from step (3)

(5) if jet axis does not move, call the trail jet a stable proto-jet, remove its constituents from the list

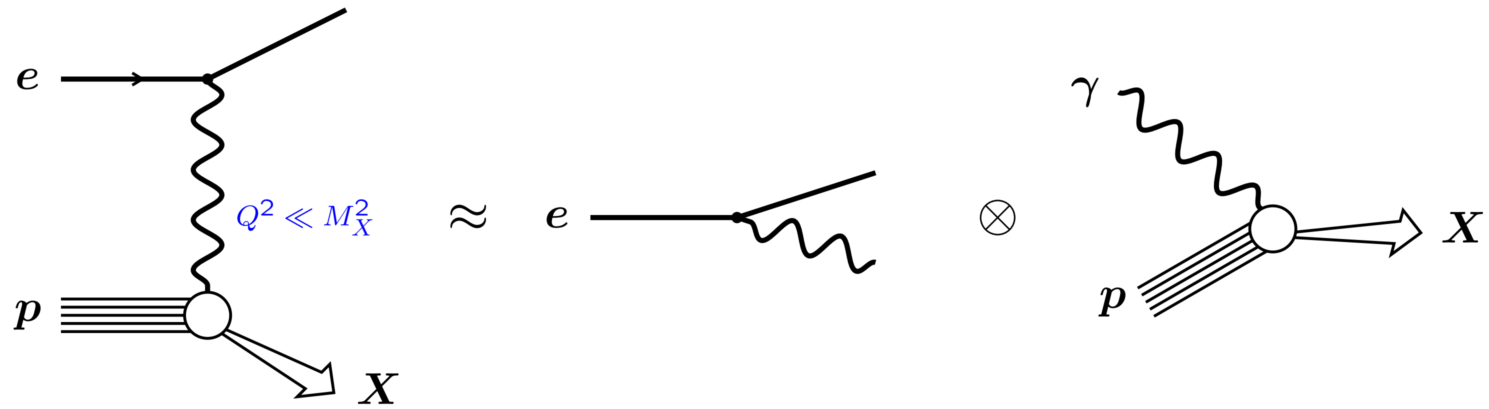
(6) repeat from (1) until no seed above certain threshold ($E_0 \sim 1$ GeV)

(7) to form final jets, split/merge overlapping proto-jets (additional parameter)

- very difficult to make cone algorithms collinear and IR safe due to seed E_0
  - DØ and CDF in Run II used Midpoint Cone algorithm (tiny change introduced by CDF ruined the IR safety)
  - seedless cone algorithm (SISCone) was too slow before development of FASTJET package
- cone algorithms have more parameters besides the cone size R
- all this makes cone algorithm theoretically unfavored

Equivalent photon approximation

Weizsäcker-Williams approximation (WWA)



- for small photon virtualities

$$e + p \to e + X$$

can be described as radiation of collinear photon and subsequent interaction of *real* photon with proton

- in proton rest frame

$$\mathrm{d}\sigma_{\mathrm{ep}} = \frac{(2\pi)^4}{|\vec{v}|}\frac{1}{2E\,2M}\frac{e^4}{Q^4}L^{\mu\nu}W_{\mu\nu}(q,p)\frac{\mathrm{d}^3\vec{k}'}{(2\pi)^3 2E_{k'}}$$

neglecting electron mass, $|\vec{v}| = 1$

- process $\gamma + p \to X$

$$\sigma_{\gamma\mathrm{p}} = \frac{(2\pi)^4 e^2}{2E_\gamma 2M}\left(-\frac{1}{2}g^{\mu\nu}\right)W_{\mu\nu}(q^2=0, q\cdot p)$$

- unpolarized e+p scattering and parity conservation

$$W_{\mu\nu} = C_1(q^2, q\cdot p)g_{\mu\nu} + C_2(q^2, q\cdot p)p_\mu p_\nu + C_3(q^2, q\cdot p)q_\mu q_\nu + C_4(q^2, q\cdot p)(p_\mu q_\nu + p_\nu q_\mu)$$

- current conservation ($J^\mu q_\mu = 0$)

$$q^\mu W_{\mu\nu} = 0 \quad \Rightarrow \quad q_\nu\left[C_1 + C_3\, q^2 + C_4\, p\cdot q\right] + p_\nu\left[C_2\, p\cdot q + C_4\, q^2\right] = 0 \quad \Rightarrow$$

$$C_4 = -C_1\frac{1}{p\cdot q} - C_3\frac{q^2}{p\cdot q}$$

$$C_2 = C_1\frac{q^2}{(p\cdot q)^2} + C_3\frac{(q^2)^2}{(p\cdot q)^2}$$

Equivalent photon approximation

- assuming that C_i are regular at Q^2 (finite cross sections)

$$\left(-\frac{1}{2}g^{\mu\nu}\right)W_{\mu\nu}(q^2=0,q\cdot p) = -2C_1 - \underbrace{\frac{1}{2}C_2M^2}_{\to 0} - \underbrace{\frac{1}{2}C_3q^2}_{\to 0} - \underbrace{C_4\,p\cdot q}_{\to -C_1} = -C_1(q^2=0)$$

- $L^{\mu\nu}q_\mu = 0$, only C_1 and C_2 terms contribute $\quad L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^\nu] = 2(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k' - m^2)g^{\mu\nu})$

$$L^{\mu\nu}W_{\mu\nu} = 2\left[-2C_1\left(k\cdot k'-2m^2\right) + C_2(2(k\cdot p)(k'\cdot p) - (k\cdot k'-m^2)M^2)\right]$$

$$Q^2 = -(k-k')^2 = 2k\cdot k' - 2m^2, \qquad y = \frac{q\cdot p}{k\cdot p} \quad\Rightarrow\quad 1-y = \frac{k'\cdot p}{k\cdot p}$$

$$L^{\mu\nu}W_{\mu\nu} = 2\left[-C_1Q^2 + 2C_1m^2 + \left(-C_1\frac{Q^2}{(p\cdot q)^2} + C_3\frac{Q^4}{(p\cdot q)^2}\right)\left(2(k\cdot p)(k'\cdot p) - M^2Q^2/2\right)\right]$$

- keeping only leading terms in $1/Q^2$, $Q^2 \ll (p.q)^2$, but $M_X^2 = s_{\gamma p} = (p+q)^2 \approx 2p\cdot q$
- also Q^2 could be as small as m^2, and we need to keep certain terms with electron mass

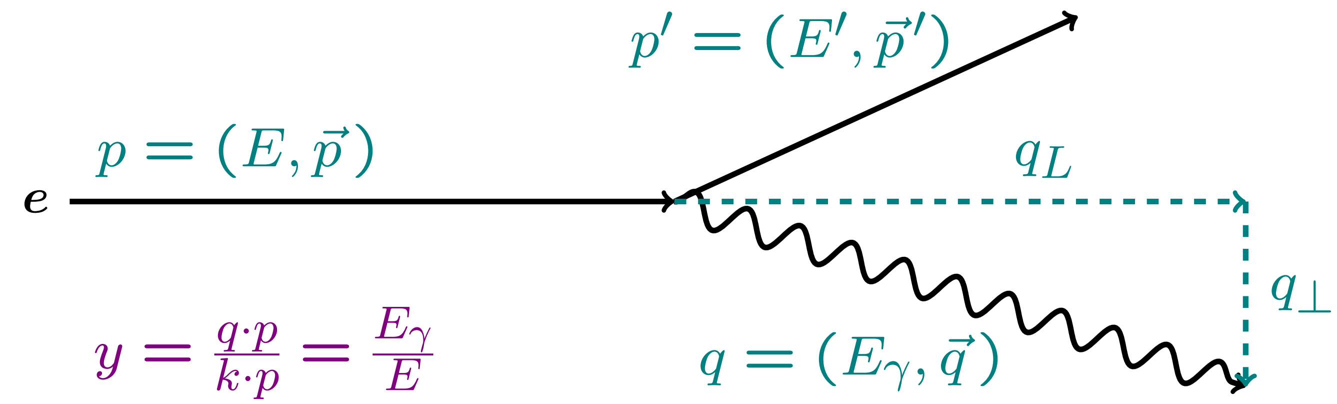
$$L^{\mu\nu}W_{\mu\nu} = -2C_1Q^2\left[1 - \frac{2m^2}{Q^2} + \frac{2(1-y)}{y^2} + \mathcal{O}(Q^2)\right] \approx -2C_1Q^2\left[\frac{1+(1-y)^2}{y^2} - \frac{2m^2}{Q^2}\right]$$

- in proton rest frame, y represents fractional energy carried away by the exchanged virtual photon

WWA - kinematics

- relation between Q^2 and y, $q_\perp$, m in the lowest order of m^2 and $q_\perp^2$ under assumptions

$$m \ll E \quad \text{and} \quad q_\perp \ll q_L$$



1) $|\vec{p}| = \sqrt{E^2 - m^2} = E(1 - m^2/E^2)^{1/2} \approx E - \frac{m^2}{2E}$

2) momentum conservation $|\vec{p}| = p'_L + q_L$ gives $|\vec{p}\,'|^2 = p_L'^2 + p_\perp'^2 = (|\vec{p}| - q_L)^2 + p_\perp'^2 = (|\vec{p}| - q_L)^2 + q_\perp^2 \quad \Rightarrow$

$$E'^2 = |\vec{p}\,'|^2 + m^2 = |\vec{p}|^2 - 2|\vec{p}|q_L + q_L^2 + q_\perp^2 + m^2 \overset{1)}{\approx} E^2 - 2Eq_L + \frac{m^2}{E}q_L + q_L^2 + q_\perp^2 = (E - q_L)^2 + \frac{q_L}{E}m^2 + q_\perp^2$$

$$\Rightarrow E' \approx (E - q_L) + \frac{1}{2}\frac{1}{E - q_L}\left(\frac{q_L}{E}m^2 + q_\perp^2\right) \qquad \Rightarrow \qquad E_\gamma = E - E' \approx q_L - \frac{1}{2}\frac{1}{E - q_L}\left(\frac{q_L}{E}m^2 + q_\perp^2\right)$$

$(E - q_L)$ is leading term only for $q_L/E \not\to 1$ (additional assumption)

3) $y = E_\gamma/E = q_L/E + \mathcal{O}(m^2, q_\perp^2)$

4) for the photon virtuality we then find

$$Q^2 = q_\perp^2 + q_L^2 - E_\gamma^2 = q_\perp^2 + (q_L - E_\gamma)(q_L + E_\gamma) \approx q_\perp^2 + \frac{1}{2}\frac{1}{E - q_L}\left(\frac{q_L}{E}m^2 + q_\perp^2\right)\left[2E_\gamma + \mathcal{O}(m^2, q_\perp^2)\right]$$

$$= q_\perp^2 + \frac{y}{1 - y}\left(m^2 y + q_\perp^2\right) = \frac{1}{1 - y}q_\perp^2 + \frac{y^2}{1 - y}m^2$$

WWA - phase space

- transform to cylindrical coordinates $\rightarrow (y, q_\perp) \rightarrow (y, Q^2)$

$Q^2 \approx \frac{1}{1-y} q_\perp^2 + \frac{y^2}{1-y} m^2$

$$\frac{\mathrm{d}^3\vec{k}'}{2E'} \xrightarrow{\vec{k}=\vec{k}'+\vec{q}} \frac{\mathrm{d}^3\vec{q}}{2E'} = \frac{\mathrm{d}q_L q_\perp \mathrm{d}q_\perp \mathrm{d}\varphi}{2(E-E_\gamma)} = \frac{\mathrm{d}q_L q_\perp \mathrm{d}q_\perp \mathrm{d}\varphi}{2E(1-y)} \approx \frac{\mathrm{d}y \mathrm{d}q_\perp^2 \mathrm{d}\varphi}{4(1-y)} \approx \frac{\mathrm{d}y \mathrm{d}Q^2 \mathrm{d}\varphi}{4}$$

- integration over azimuthal angle gives phase space factor

$$\frac{\pi}{2}\mathrm{d}y\mathrm{d}Q^2$$

- we are ready to put things together

$$\sigma_{\gamma \mathrm{p}} = \frac{(2\pi)^4 e^2}{2E\,2M}\frac{E}{E_\gamma}(-C_1(q^2=0)) = -C_1\frac{(2\pi)^4 e^2}{2E\,2M}\frac{1}{y}$$

$$\mathrm{d}\sigma_{\mathrm{ep}} = \frac{(2\pi)^4}{2E\,2M}\frac{e^4}{Q^4}(-2C_1 Q^2)\left[\frac{1+(1-y)^2}{y^2} - \frac{2m^2}{Q^2}\right]\frac{1}{(2\pi)^3}\frac{\pi}{2}\mathrm{d}y\mathrm{d}Q^2$$

$$\mathrm{d}\sigma_{\mathrm{ep}} = y\sigma_{\gamma\mathrm{p}} \cdot \frac{e^2}{8\pi^2}\left[\frac{1+(1-y)^2}{y^2} - \frac{2m^2}{Q^2}\right]\mathrm{d}y\frac{\mathrm{d}Q^2}{Q^2} = \sigma_{\gamma\mathrm{p}}(s_{\gamma\mathrm{p}}) \cdot \frac{\alpha}{2\pi}\left[\frac{1+(1-y)^2}{y} - \frac{2m^2 y}{Q^2}\right]\mathrm{d}y\frac{\mathrm{d}Q^2}{Q^2}$$

- expression $\frac{\alpha}{2\pi}\left[\frac{1+(1-y)^2}{y} - \frac{2m^2 y}{Q^2}\right]\mathrm{d}y\frac{\mathrm{d}Q^2}{Q^2}$ can be interpreted as **probability of finding photon inside electron** with fractional energy in interval $(y, y+\mathrm{d}y)$ and with virtuality in interval $(Q^2, Q^2+\mathrm{d}Q^2)$

Splitting functions in QED

$Q^2 \approx \frac{1}{1-y} q_\perp^2 + \frac{y^2}{1-y} m^2$

- integration from $Q^2_{\text{min}} = m^2 y^2/(1-y)$ to Q^2_{max} leads to

$$\int_{Q^2_{\text{min}}}^{Q^2_{\text{max}}} \mathrm{d}Q^2 \frac{1}{Q^2}\left[\frac{1+(1-y)^2}{y} - \frac{2m^2 y}{Q^2}\right] = \frac{1+(1-y)^2}{y} \ln\frac{Q^2_{\text{max}}(1-y)}{m^2y^2} - 2m^2 y\left[\frac{1}{Q^2_{\text{min}}} - \frac{1}{Q^2_{\text{max}}}\right]$$
$$\to \frac{1+(1-y)^2}{y} \ln\frac{Q^2_{\text{max}}(1-y)}{m^2y^2} - \frac{2(1-y)}{y}$$

- mass singularity is still there, as $m \to 0 \;\Rightarrow\; Q^2_{\text{min}} = y^2 m^2/(1-y) \to 0$ leading to infinite cross section
- term $\frac{1+(1-y)^2}{y}$ is dominant (leading singularity for $m \to 0$)
- the mass term $2m^2 y/Q^2$ still gives finite contribution even in limit $m \to 0$
- splitting function - the term in front of leading divergence describing the probability of radiation

$$P_{\gamma e}(y) = \frac{1+(1-y)^2}{y} \qquad \Rightarrow \qquad P_{ee}(y) = P_{\gamma e}(1-y) = \frac{1+y^2}{1-y}$$

- WWA - example of regularization using fermion mass rather than gluon (photon) mass
- documents the universality of splitting functions (compare with $P_{qq}(z) = C_F \frac{1+z^2}{1-z}$ in QCD)

For reading and for exercises

- For reading (J. Chyla textbook)
  - chapter 9.3 (jets)
  - chapter 10.2.1 (Equivalent photon approximation)
- Exercise
  - exercise 7.3 (3-jet cross section in e^+e^-)
  - exercise 8.3 (3-jet cross section in collinear approximation)