Quarks, Partons, and Quantum Chromodynamics

Lecture 11

Mass singularities in QCD

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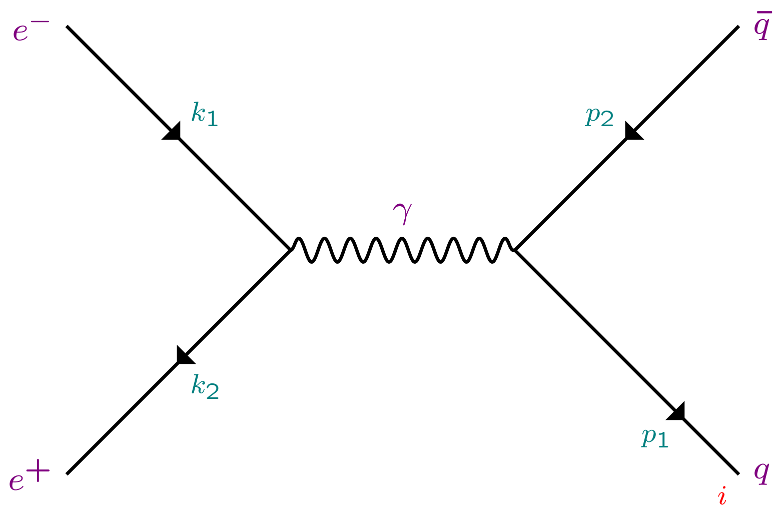
Mass singularities

- gauge theories, like QED or QCD, have besides UV divergences (short distance behavior) another type of divergences connected with large distance scales
- they appear if we have massless particles (or if we neglect their masses)
- the treatment is different
  - we only need to regularize them (no renormalization)
  - if we define carefully the final state (we put physically indistinguishable states together) we obtain finite cross sections
  - stand-alone massless e^- cannot be distinguished from the state when electron radiates photon in the direction of its flight

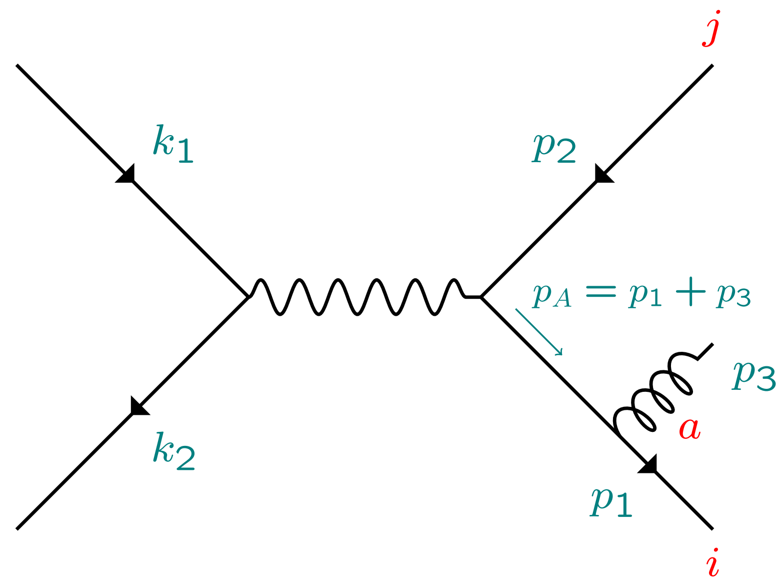
  - massive e^- is slower than foton and they are separated and distinguishable (at least in principle)

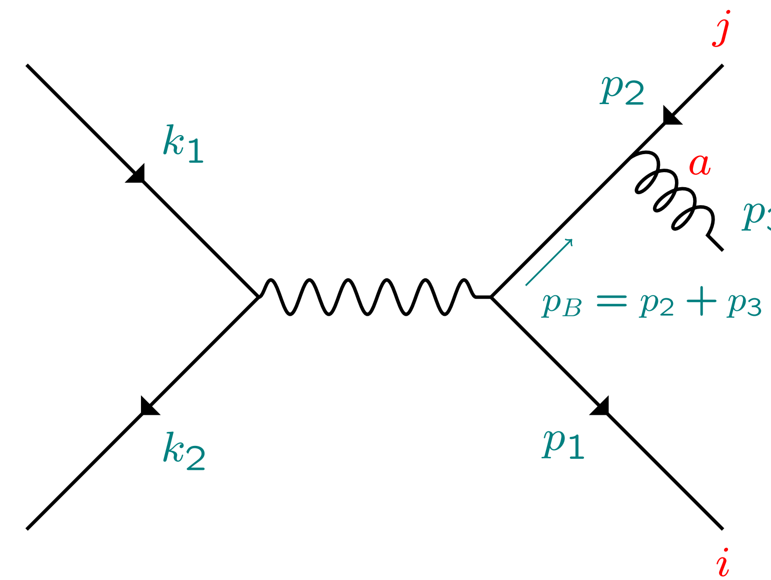
$e^+e^- \to$ hadrons

LO - $e^+e^- \to q\bar{q}$



next order - $e^+e^- \to q\bar{q}g$



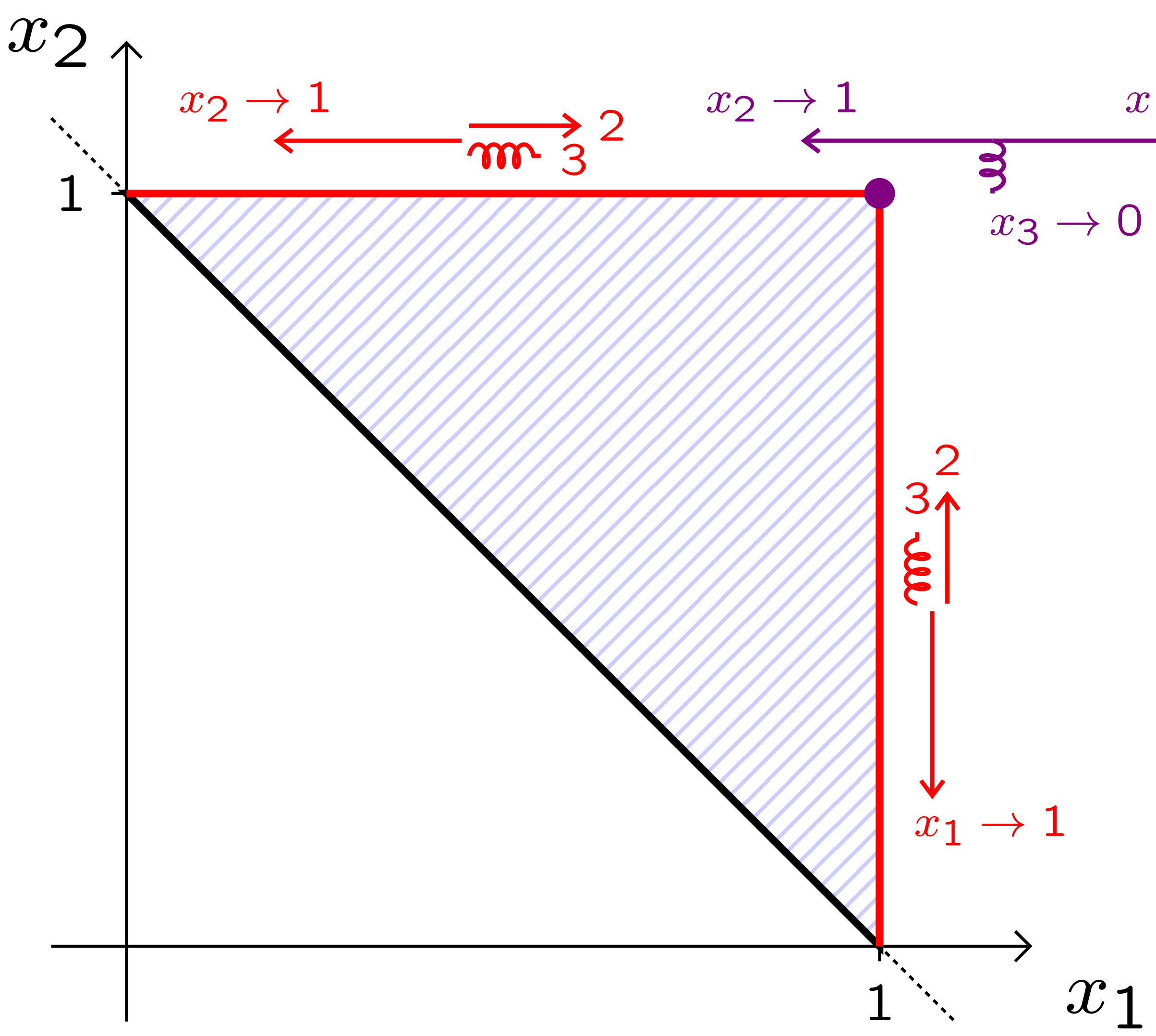


$$\sigma_0 = 3\frac{4\pi\alpha^2}{3s}e_q^2$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = C_F\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)}$$

- $\sigma_{\text{real}} \equiv \int \frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} \to \infty$
- real gluon emission – only part of NLO QCD

Mass singularities in $e^+e^- \to q\bar{q}g$



$$x_i \equiv \frac{2E_i}{\sqrt{s}}, \quad 1\text{ - quark, } 2\text{ - antiquark, } 3\text{ - gluon}$$

- energy conservation $x_1 + x_2 + x_3 = 2$
- momentum conservation $(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0)$

$$\Rightarrow \; 0 < x_i < 1$$

- **collinear singularities** $(x_1 \to 1$ or $x_2 \to 1)$
- **IR singularities** (both $x_1 \to 1$ and $x_2 \to 1$)
  - implies soft gluon $(x_3 \to 0)$

$$\sigma_{\text{real}} = C_F \frac{\alpha_S}{2\pi} \sigma_0 \int_0^1 \mathrm{d}x_1 \int_{1-x_1}^1 \mathrm{d}x_2 \, \frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)} \to \infty$$

$$p_1 + p_2 + p_3 = q(=(Q,\vec{0})) \;\Rightarrow\; m_{23}^2 = (p_2+p_3)^2 = (q-p_1)^2 = Q^2 - 2QE_1 = Q^2(1-x_1) \;\Rightarrow\; y_{ij} \equiv \frac{m_{ij}^2}{s} = (1-x_k)$$

$$\frac{\mathrm{d}\sigma_{\text{real}}}{\mathrm{d}y_{13}\,\mathrm{d}y_{23}} = C_F \frac{\alpha_S}{2\pi} \sigma_0 \frac{(1-y_{23})^2 + (1-y_{13})^2}{y_{13}\, y_{23}}$$

- singular when one of qg or $\bar{q}g$ invariant masses approaches zero
- this is possible only if $m_q = m_g = 0 \Rightarrow$ **mass singularities**

$q \to qg$ splitting functions $P_{qq}(z)$ and $P_{gq}(z)$

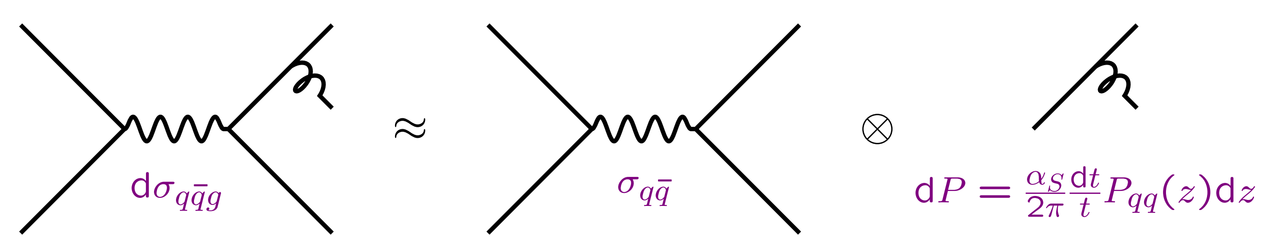
$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = C_F\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)} \;\Rightarrow\; \frac{\mathrm{d}\sigma}{\mathrm{d}m_{13}^2\mathrm{d}x_1} = \frac{1}{Q^2}\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = C_F\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{\underbrace{Q^2(1-x_2)}_{m_{13}^2\;\ldots\;\text{virtuality of virtual } q}(1-x_1)} = C_F\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{m_{13}^2(1-x_1)}$$

- dominant contribution from kinematic region $x_2 \to 1$ (and similarly for $x_1 \to 1$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}m_{13}^2\mathrm{d}x_1} \stackrel{x_2\to 1}{\to} \sigma_0\frac{\alpha_S}{2\pi}\frac{1}{m_{13}^2}\underbrace{C_F\frac{1+x_1^2}{1-x_1}}_{P_{qq}(x_1)\;\ldots\;\textbf{splitting function}}$$

- in this kinematic domain we can say that gluon is radiated from quark
- splitting function $P_{qq}(z)$ describes the distribution of quark fractional energy after radiation of gluon
- universal - the leading (divergent) term does not depend on the origin of quark (ee, DIS, ...)
- factorization - first we create quark ($ee \to q\bar{q}$) and then we let it to radiate with probability

$$\mathrm{d}P = \frac{\alpha_S}{2\pi}\frac{\mathrm{d}t}{t}P_{qq}(z)\mathrm{d}z$$



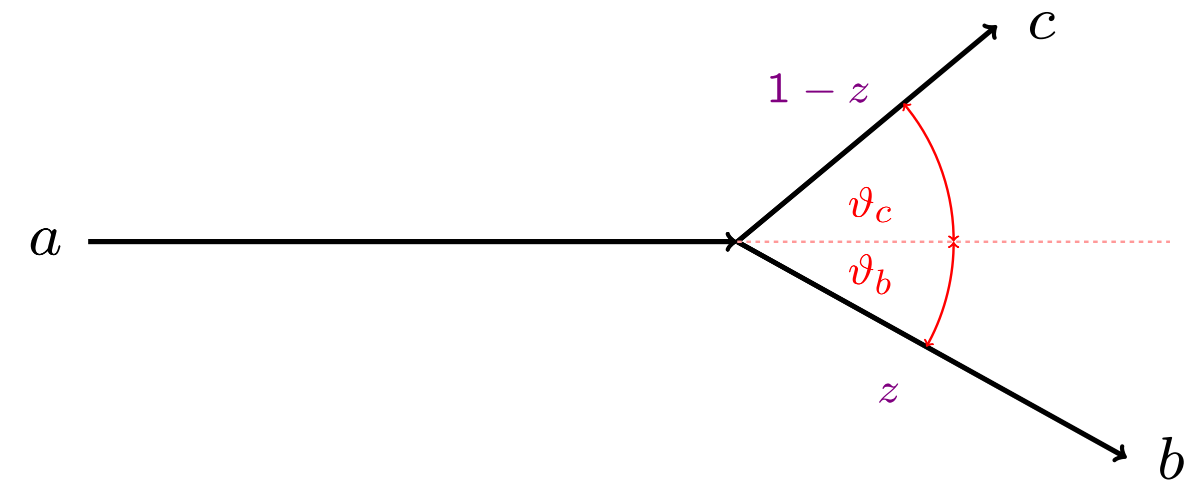
- z is fractional energy of quark $\Rightarrow$ gluon carries $1-z \Rightarrow$ $P_{gq}(z) = P_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z}$

Splitting functions in QCD (ex. 8.1)

based on Ellis, Stirling, Webber: QCD and collider physics

- universality $\Rightarrow$ the splitting functions can be derived from 2 body decay of virtual parton in collinear limit $\vartheta = \vartheta_b + \vartheta_c \to 0$

$a \to b\, c$



Kinematics

- energy conservation $E_a = E_b + E_c$
- z - fractional energy carried by parton b

$$E_b = zE_a, \qquad E_c = (1-z)E_a$$

- momentum conservation in transversal direction

$$z \sin\vartheta_b = (1-z)\sin\vartheta_c \quad \xrightarrow{\vartheta \ll 1} \quad z(\vartheta_b + \vartheta_c) = \vartheta_c \quad \Rightarrow \quad \vartheta_c = z\vartheta \quad \text{and} \quad \vartheta_b = (1-z)\vartheta$$

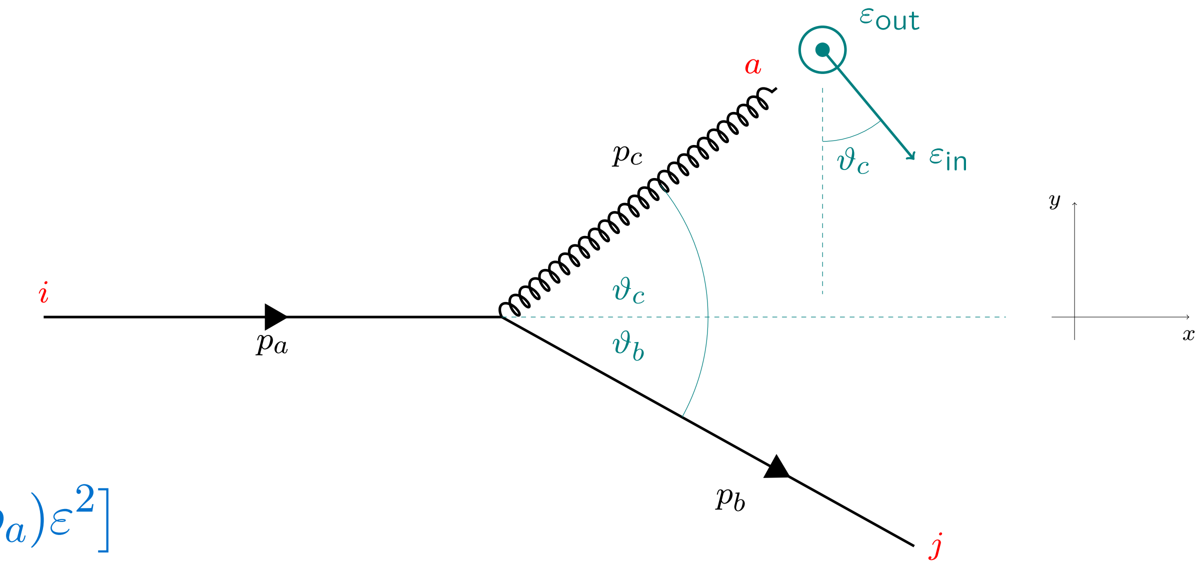
- invariant mass of bc pair (and virtuality of decaying parton a)

$$t = 2E_bE_c(1-\cos\vartheta) \quad \xrightarrow{\vartheta \ll 1} \quad E_a^2\vartheta^2 z(1-z)$$

Splitting function $P_{qq}(z)$

Matrix element for process $q \to qg$

$$i\mathcal{M} = i\bar{u}(p_b, s_b) i g \gamma^\mu T^a_{ji} u(p_a, s_a) \varepsilon_\mu(p_c)$$



Squared, averaged over initial states, and summed over final state spins and colors, except for gluon polarization

$$|\mathcal{M}|^2 = \underbrace{\frac{1}{2}\frac{1}{3}\mathrm{Tr}(T^a T^a)}_{C_F} g^2 \mathrm{Tr}(\not{p}_b \not{\varepsilon} \not{p}_a \not{\varepsilon}) = 2g^2 C_F \left[2(p_b \cdot \varepsilon)(p_a \cdot \varepsilon) - (p_b \cdot p_a)\varepsilon^2\right]$$

Normalization of polarization vector ε^μ is fixed by condition $\varepsilon^2 = -1$. The two transversal polarization states can be picked in such a way, that one polarization vector is in the plane of the decay products (ε_{in}) while the second one is perpendicular to this plane (ε_{out}). The coordinates of the two polarization vectors are $\varepsilon_{\text{in}} = (0, \vartheta_c, -1, 0)$ and $\varepsilon_{\text{out}} = (0, 0, 0, -1)$.

The signs in polarization vectors coordinates were chosen in such a way that vectors $\vec{p_c}$, $\vec{\varepsilon}_{\text{in}}$ and $\vec{\varepsilon}_{\text{out}}$ constitute a right-handed system similar to axis x, y, and z. This will be important later on, when we analyze $g \to gg$ case.

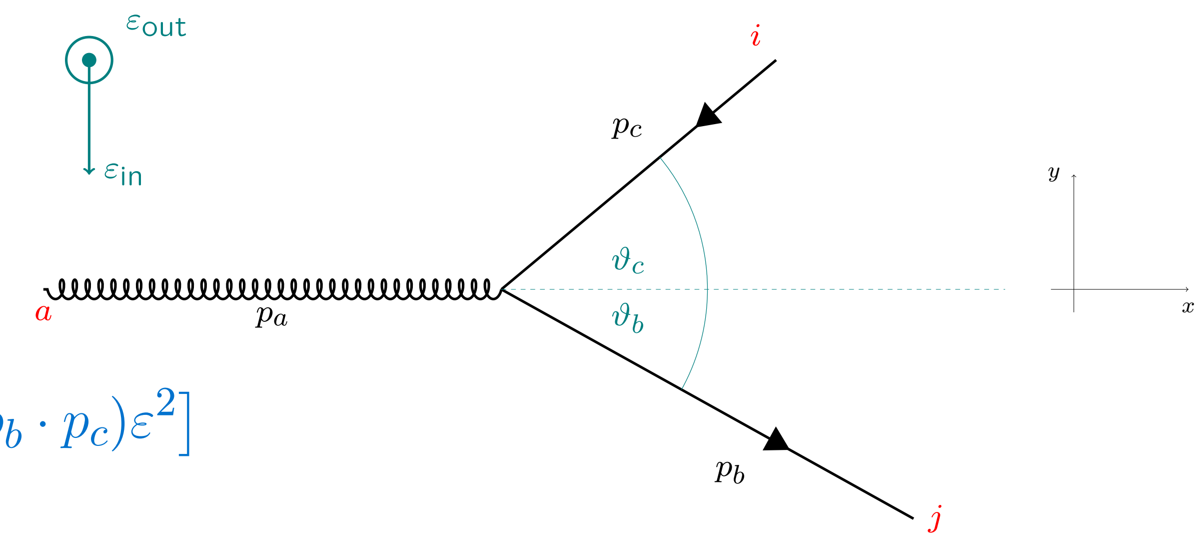
Quark 4-momenta read $p_a = E_a(1, 1, 0, 0)$ and $p_b = zE_a(1, 1 - \vartheta_b^2/2, -\vartheta_b, 0)$, where we have expanded the trigonometric functions up to the relevant terms in angle ϑ. We neglected the virtuality of decaying quark, slightly violating 4-momentum conservation, hoping that this will not affect the final result. Note also that $p_a \cdot \varepsilon_{\text{out}} = p_b \cdot \varepsilon_{\text{out}} = 0$.

$$|\mathcal{M}|^2 = 2g^2 C_F E_a^2 z \left[2(\vartheta_c + \vartheta_b)\vartheta_c + \vartheta_b^2\right] = 2g^2 C_F E_a^2 \vartheta^2 z \left[2z + (1-z)^2\right] = 2g^2 C_F E_a^2 \vartheta^2 z (1-z) \frac{1+z^2}{1-z} = 2g^2 t P_{qq}(z)$$

Splitting function $P_{qg}(z)$

Matrix element for process $g \to q\bar{q}$

$$i\mathcal{M} = \bar{u}(p_b, s_b) ig T^a_{ji} \gamma^\mu u(p_c, s_c) \varepsilon^\star_\mu(p_a)$$



Squared, averaged and summed, except for gluon polarizations

$$|\mathcal{M}|^2 = \underbrace{\frac{1}{8}\mathrm{Tr}(T^a T^a)}_{\frac{1}{8}T_R\delta_{aa} = T_R (=1/2)} \; g^2 \mathrm{Tr}(\not{p}_b \not{\varepsilon} \not{p}_c \not{\varepsilon}) = 4 T_R g^2 \left[2(p_b \cdot \varepsilon)(p_c \cdot \varepsilon) - (p_b \cdot p_c)\varepsilon^2 \right]$$

Now we have

$$\varepsilon_{\text{in}} = (0, 0, -1, 0), \quad \varepsilon_{\text{out}} = (0, 0, 0, -1), \quad p_b = z E_a(1, 1 - \vartheta_b^2/2, -\vartheta_b, 0), \quad p_c = (1-z) E_a(1, 1 - \vartheta_c^2/2, \vartheta_c, 0)$$

Matrix element averaged over the two initial gluon polarizations

$$|\mathcal{M}|^2 = \frac{1}{2} 4 T_R g^2 E_a^2 z(1-z) \left[-2\vartheta_b\vartheta_c + \vartheta_b^2 + \vartheta_c^2 + 2\vartheta_b\vartheta_c \right] = 2g^2 E_a^2 \vartheta^2 z(1-z) T_R \left[z^2 + (1-z)^2 \right] = 2g^2 t T_R \left[z^2 + (1-z)^2 \right]$$

Splitting function for finding quark inside gluon is then

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

The splitting function for finding anti-quark inside gluon, is due to symmetry of $P_{qg}(z) = P_{qg}(1-z)$,

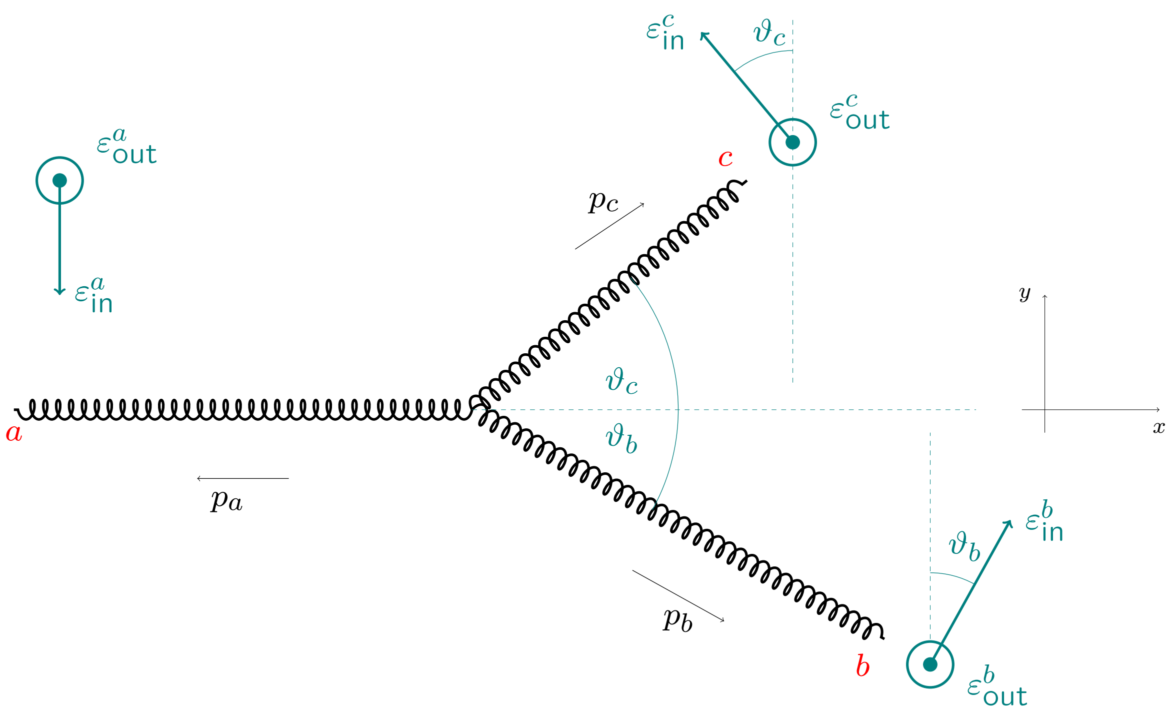
$$P_{\bar{q}g}(z) = P_{qg}(1-z) = P_{qg}(z)$$

Splitting function $P_{gg}(z)$

Matrix element for process $g \to gg$ (all momenta outgoing)

$$i\mathcal{M} = g f_{abc} C^{\mu\nu\lambda}(p_a, p_b, p_c)\varepsilon^\star_\mu(p_a)\varepsilon_\nu(p_b)\varepsilon_\lambda(p_c)$$

$$C^{\mu\nu\lambda}(p_a, p_b, p_c) = g^{\mu\nu}(p_a - p_b)^\lambda + g^{\nu\lambda}(p_b - p_c)^\mu + g^{\lambda\mu}(p_c - p_a)^\nu$$



Color factor

$$\frac{1}{8} f_{abc} f_{abc} = -\frac{1}{8} f_{abc} f_{acb} = \frac{1}{8}\mathrm{Tr}(F^a F^a) = \frac{1}{8} C_A \mathrm{Tr}\,\mathbf{1}_8 = C_A$$

Kinematic factor

$$K = C^{\mu\nu\lambda}(p_a, p_b, p_c)\varepsilon^\star_\mu(p_a)\varepsilon_\nu(p_b)\varepsilon_\lambda(p_c) = (\varepsilon_a\cdot\varepsilon_b)\left[(p_a - p_b)\cdot\varepsilon_c\right] + (\varepsilon_b\cdot\varepsilon_c)\left[(p_b - p_c)\cdot\varepsilon_a\right] + (\varepsilon_a\cdot\varepsilon_c)\left[(p_c - p_a)\cdot\varepsilon_b\right]$$

can be rewritten using 4-momentum conservation ($p_a + p_b + p_c = 0$) and transversality of polarization, $q\cdot\varepsilon(q) = 0$,

$$K = 2(p_a\cdot\varepsilon_c)(\varepsilon_a\cdot\varepsilon_b) + 2(p_b\cdot\varepsilon_a)(\varepsilon_b\cdot\varepsilon_c) + 2(p_c\cdot\varepsilon_b)(\varepsilon_a\cdot\varepsilon_c).$$

$$p_a = E_a(1, -1, 0, 0) \quad p_b = zE_a(1, 1 - \vartheta_b^2/2, -\vartheta_b, 0) \quad p_c = (1-z)E_a(1, 1 - \vartheta_c^2/2, \vartheta_c, 0)$$

$$\varepsilon^a_{\text{in}} = (0, 0, -1, 0) \quad \varepsilon^b_{\text{in}} = (0, \vartheta_b, 1, 0) \quad \varepsilon^c_{\text{in}} = (0, -\vartheta_c, 1, 0) \qquad \varepsilon^a_{\text{out}} = \varepsilon^b_{\text{out}} = \varepsilon^c_{\text{out}} = (0, 0, 0, 1)$$

$$p_a\cdot\varepsilon^c_{\text{in}} = -E_a\vartheta_c = -zE_a\vartheta$$

$$p_b\cdot\varepsilon^a_{\text{in}} = -zE_a\vartheta_b = -z(1-z)E_a\vartheta$$

$$p_c\cdot\varepsilon^b_{\text{in}} = -(1-z)E_a(\vartheta_b + \vartheta_c) = -(1-z)E_a\vartheta$$

$$\varepsilon^a_{\text{in}}\cdot\varepsilon^b_{\text{in}} = \varepsilon^a_{\text{in}}\cdot\varepsilon^c_{\text{in}} = 1$$

$$\varepsilon^b_{\text{in}}\cdot\varepsilon^c_{\text{in}} = -1$$

$$p\cdot\varepsilon_{\text{out}} = 0$$

$$\varepsilon_{\text{in}}\cdot\varepsilon_{\text{out}} = 0$$

Splitting function $P_{gg}(z)$

- kinematic factors for all possible polarizations

$K_{\mathsf{ooo}} = 0$ (all three gluons are polarized perpendicular to the decay plane, we obtain trivially zero)

$$K_{\mathsf{ioo}} = 2(p_b \cdot \varepsilon^a_{\mathsf{in}})(\varepsilon^b_{\mathsf{out}} \cdot \varepsilon^c_{\mathsf{out}}) = 2z(1-z)E_a\vartheta$$

$$K_{\mathsf{oio}} = 2(p_c \cdot \varepsilon^b_{\mathsf{in}})(\varepsilon^a_{\mathsf{out}} \cdot \varepsilon^c_{\mathsf{out}}) = 2(1-z)E_a\vartheta$$

$$K_{\mathsf{ooi}} = 2(p_a \cdot \varepsilon^c_{\mathsf{in}})(\varepsilon^b_{\mathsf{out}} \cdot \varepsilon^c_{\mathsf{out}}) = 2zE_a\vartheta$$

$$K_{\mathsf{oii}} = K_{\mathsf{ioi}} = K_{\mathsf{iio}} = 0$$

$$K_{\mathsf{iii}} = 2\left[-z + z(1-z) - (1-z)\right]E_a\vartheta$$

For the sum of squares we have then

$$|K_{\mathsf{i}}|^2 = |K_{\mathsf{ioo}}|^2 + |K_{\mathsf{oio}}|^2 + |K_{\mathsf{ooi}}|^2 = 4E_a^2\vartheta^2\left[z^2(1-z)^2 + (1-z)^2 + z^2\right] = 4\,\overbrace{E_a^2\vartheta^2 z(1-z)}^{t}\left[z(1-z) + \frac{1-z}{z} + \frac{z}{1-z}\right]$$

$$|K_{\mathsf{iii}}|^2 = 4E_a^2\vartheta^2\left[z^2 + z^2(1-z)^2 + (1-z)^2 - 2z^2(1-z) + 2z(1-z) - 2z(1-z)^2\right]$$

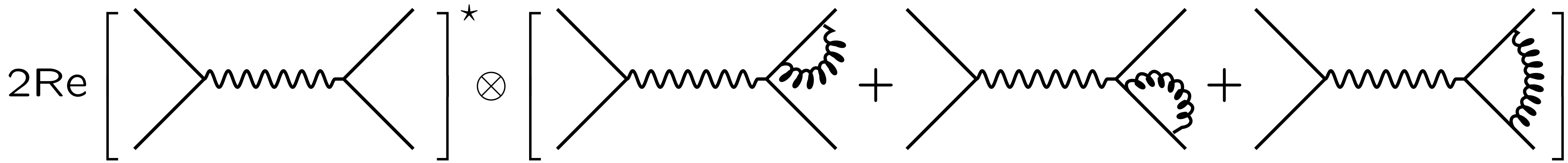
$$= |K_{\mathsf{i}}|^2 + 8E_a^2\vartheta^2 z(1-z)(\underbrace{-z+1-1+z}_{=0})$$

For matrix element we then have

$$|\mathcal{M}|^2 = \frac{1}{2}C_A g^2 8t\left[z(1-z) + \frac{1-z}{z} + \frac{z}{1-z}\right] \qquad \Rightarrow \qquad P_{gg}(z) = 2C_A\left[z(1-z) + \frac{1-z}{z} + \frac{z}{1-z}\right]$$

$e^+e^- \to$ hadrons at order of α_S

- $e^+e^- \to q\bar{q}$ at LO ($\mathcal{O}(\alpha_S^0)$) - σ_0
- $e^+e^- \to q\bar{q}g$ at LO ($\mathcal{O}(\alpha_S^1)$) - $\sigma_{\text{real}} \to +\infty$
- $e^+e^- \to q\bar{q}$ at NLO ($\mathcal{O}(\alpha_S^1)$) - interference term - $\sigma_{\text{virtual}} \to -\infty$

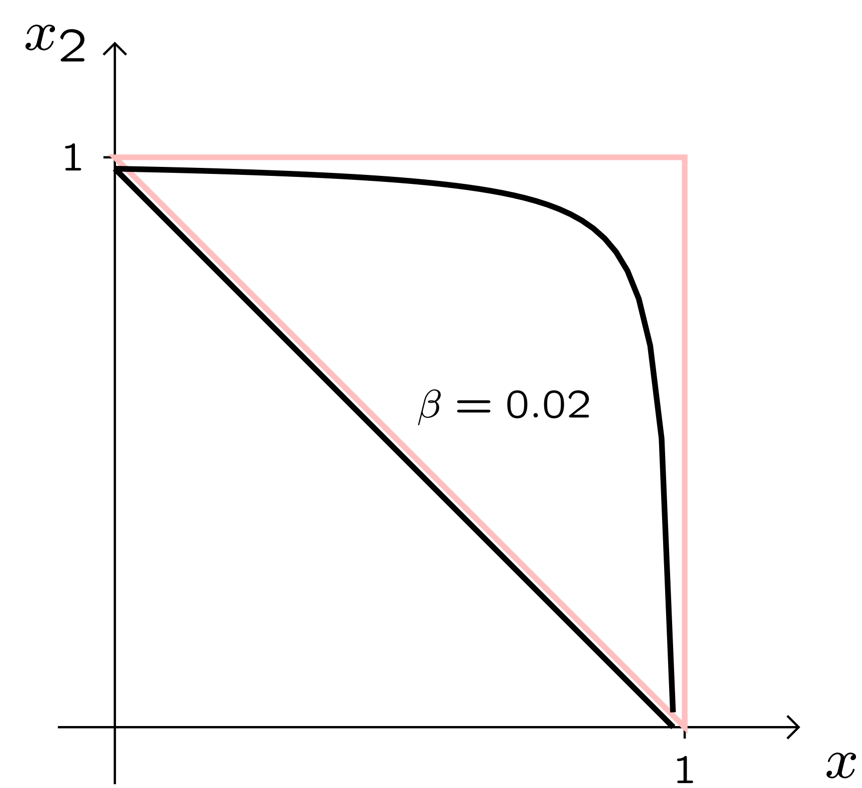


- first we need to tame the infinities - **regularization** by introducing gluon mass m_g: $\beta = m_g^2/Q^2$
  (other options are m_q, dimensional regularization, ...)
- this modifies the kinematic boundaries (ex. 7.2)

$$0 \le x_1 \le 1 - \beta, \qquad 1 - \beta - x_1 \le x_2 \le 1 - \frac{\beta}{1 - x_1}$$

and gluon propagator

$$-g_{\mu\nu} \to -g_{\mu\nu} + q_\mu q_\nu / m_g^2$$



$e^+e^- \to$ hadrons at order of α_S

R. Field: Applications of perturbative QCD

$$\frac{\mathrm{d}\sigma_{\text{real}}}{\mathrm{d}x_1\mathrm{d}x_2} \stackrel{\beta\to 0}{=} \sigma_0 \frac{\alpha_S}{2\pi} C_F \frac{1}{(1-x_2)(1-x_1)} \left(x_1^2 + x_2^2 + \beta \left[2x_1 + 2x_2 + \frac{(1-x_1)^2 + (1-x_2)^2}{(1-x_2)(1-x_1)} \right] + 2\beta^2 \right)$$

- terms proportional to β and β^2 still give finite contribution after the integration of phase space
- integration over x_2 leads to ($x \equiv x_1$)

$$\frac{\mathrm{d}\sigma_{\text{real}}}{\mathrm{d}x} \stackrel{\beta\to 0}{=} \sigma_0 \frac{\alpha_S}{2\pi} C_F \left[\frac{1+x^2}{1-x} \ln \frac{x(1-x)}{\beta} - \frac{3}{2}\frac{1}{1-x} + \frac{1}{2}(1+x) + \beta \frac{2-x}{(1-x)^2} + \frac{1}{2}\beta^2 \frac{1}{(1-x)^3} \right]$$

- note that in front of divergent term ($\ln \beta$) we find the splitting function $P_{qq}(x)$ - **universality**
- further integration over $x_1 (= x)$ leads to

$$\sigma_{\text{real}} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \left(\ln^2 \beta + 3 \ln \beta - \frac{\pi^2}{3} + 5 \right)$$

- if we do the same for the virtual contribution ($e^+e^- \to q\bar{q}$ at NLO)

$$\sigma_{\text{virt}} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \left(-\ln^2 \beta - 3 \ln \beta + \frac{\pi^2}{3} - \frac{7}{2} \right)$$

$\Rightarrow$ $\sigma_{\text{NLO}} = \sigma_0 + \sigma_{\text{real}} + \sigma_{\text{virt}} = \sigma_0(1 + \alpha_S/\pi)$ is finite even in the limit $\beta \to 0$

Virtual correction to splitting function - "+" distribution

- virtual emission at $x = 1$

$$\frac{\mathrm{d}\sigma_{\text{virt}}}{\mathrm{d}x} = \sigma_0 \frac{\alpha_S}{2\pi} C_F \left(-\ln^2\beta - 3\ln\beta + \frac{\pi^2}{3} - \frac{7}{2} \right) \delta(1-x)$$

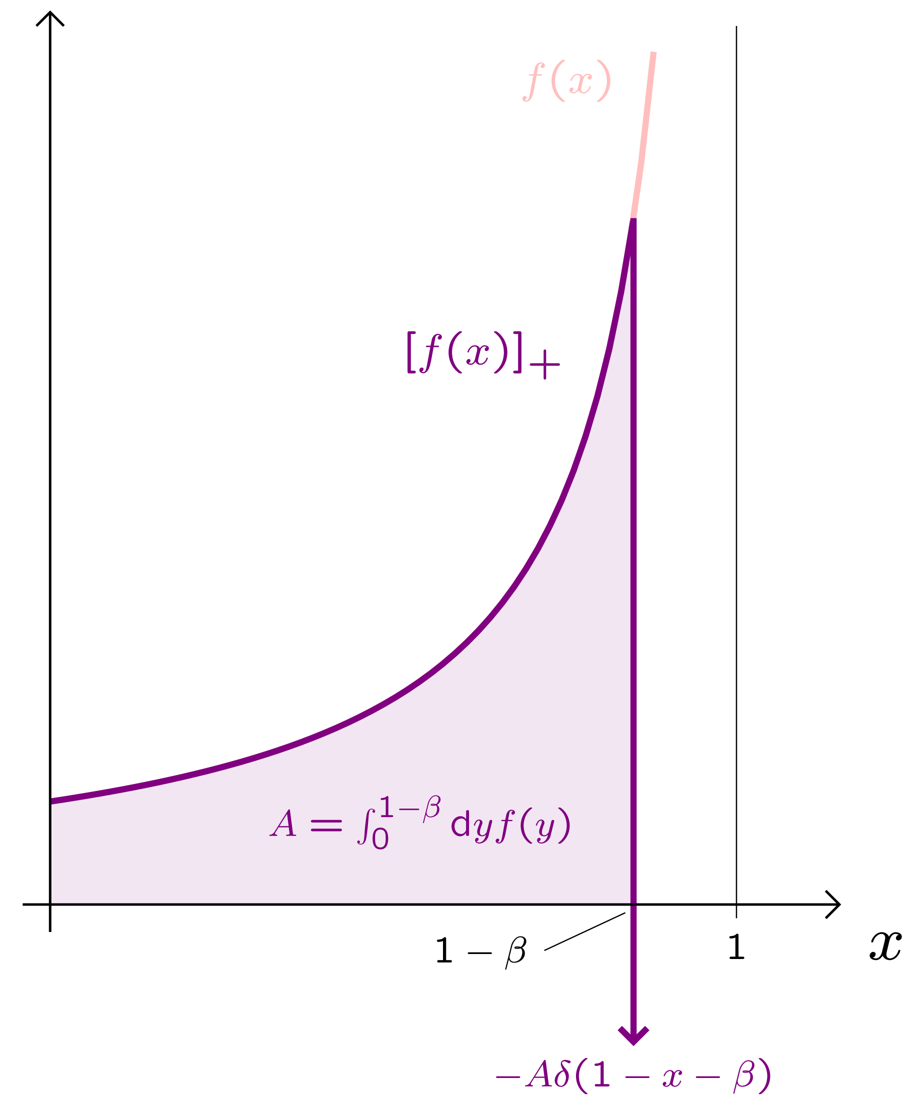
- radiation from quark is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = \frac{\mathrm{d}\sigma_{\text{real}}}{\mathrm{d}x} + \frac{\mathrm{d}\sigma_{\text{virt}}}{\mathrm{d}x} \quad \text{more precisely} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}x} = \lim_{\beta\to 0}\left[\frac{\mathrm{d}\sigma_{\text{real}}}{\mathrm{d}x}\theta(1-x-\beta) + \sigma_{\text{virt}}\delta(1-x-\beta)\right]$$

- concept of **"+"-distribution**

$$[f(x)]_+ \equiv \lim_{\beta\to 0}\left[f(x)\theta(1-x-\beta) - \delta(1-x-\beta)\int_0^{1-\beta}\mathrm{d}y\, f(y)\right]$$

from the definition: $\displaystyle\int_0^1 \mathrm{d}x [f(x)]_+ = 0$



- divergent part ($\ln\beta$ term) in real emission must vanish after adding virtual emission and integration

$$P_{qq}(x) = C_F\frac{1+x^2}{1-x} \to C_F\left[\frac{1+x^2}{1-x}\right]_+ + K\delta(1-x)$$

- $K = 0$ from explicit calculation (or from neat trick)

Virtual correction to splitting function - "+" distribution

$$\frac{d\sigma}{dx} = \lim_{\beta\to 0}\left[\frac{d\sigma_{\text{real}}}{dx}\theta(1-x-\beta) + \sigma_{\text{virt}}\delta(1-x-\beta)\right]$$
$$= \lim_{\beta\to 0}\sigma_0\frac{\alpha_S}{2\pi}C_F\left[\left(\frac{1+x^2}{1-x}\ln\frac{x(1-x)}{\beta} - \frac{3}{2}\frac{1}{1-x} + \frac{1}{2}(1+x) + \beta\frac{2-x}{(1-x)^2} + \frac{1}{2}\beta^2\frac{1}{(1-x)^3}\right)\theta(1-x-\beta)\right.$$
$$\left. - \left(-\ln^2\beta - 3\ln\beta + \frac{\pi^2}{3} - \frac{7}{2}\right)\delta(1-x-\beta)\right]$$

$A)\ \beta\frac{2-x}{(1-x)^2} \to \delta(1-x)$

$B)\ \frac{1}{2}\beta^2\frac{1}{(1-x)^3} \to \frac{1}{4}\delta(1-x)$

$C)\ \left[\frac{1}{1-x}\right]_+ = \lim_{\beta\to 0}\left[\frac{1}{1-x}\theta(1-x-\beta) + \ln\beta\,\delta(1-x-\beta)\right]$

$D)\ \left[\frac{1+x^2}{1-x}\right]_+ = (1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\delta(1-x)$

$E)\ (1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+ = \lim_{\beta\to 0}\left[\frac{1+x^2}{1-x}\ln(1-x)\theta(1-x-\beta) + \ln^2\beta\,\delta(1-x-\beta)\right]$

Prove them!

$$\frac{d\sigma}{dx} = \sigma_0\frac{\alpha_S}{2\pi}C_F\left\{\ln\frac{1}{\beta}\left[\frac{1+x^2}{1-x}\right]_+ + (1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+ + \frac{1+x^2}{1-x}\ln x - \frac{3}{2}\left[\frac{1}{1-x}\right]_+ + \frac{1+x}{2} + \left(\frac{\pi^2}{3} - \frac{9}{4}\right)\delta(1-x)\right\}$$

leading, divergent term

constant terms

$\Rightarrow$ virtual corrections modify $P_{qq}(x)$ to $\quad P_{qq}(x) = C_F\left[\frac{1+x^2}{1-x}\right]_+$

KLN theorem

Kinoshita-Lee-Naunberg

Consider the general scattering process $A \to B$ between states A and B of massive particles. It may happen that the scattering matrix $|S_{AB}|$ has mass singularities but if we sum the squares of scattering amplitudes

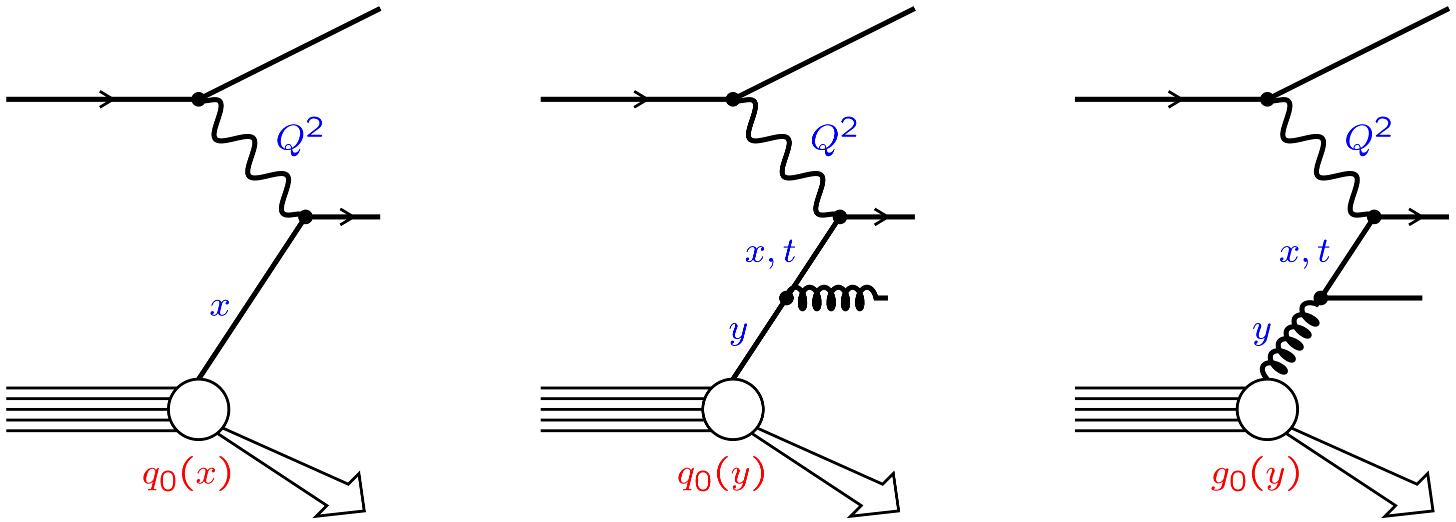
$$\sum_{D(A),D(B)} |S_{D(A),D(B)}|^2$$

over the sets $D(A)$, $D(B)$ of states degenerate with A and B, the sum has no mass singularities, i.e. is finite even for massless particles.

- **degenerate states** - all states of particles that have the same conserving quantum numbers as well as the same total four-momentum — experimentally cannot be distinguished

- as electron and quarks have masses, only γ/g mass singularities are present in QED/QCD
- however $m_q \ll \Lambda_{\text{QCD}} \ll Q_{\text{process}}$ + confinement + experimental resolution is usually worse than m_q
- the KLN theorem and techniques how to deal with mass singularities very useful in QCD
- we cannot characterize the final state by number of partons - to get more detailed description of final state (besides some global variables as event shapes) we need to introduce new concepts that put all degenerate states into one object - **jets**

PDFs in QCD



$$\mathrm{d}P = \frac{\alpha_S}{2\pi}\frac{\mathrm{d}t}{t}P_{qq}(z)\mathrm{d}z$$

- QPM - partons are quasi-free
  $\Rightarrow$ on mass shell (no virtuality)
  proton described by **bare** PDF $q_0(x)$ and $g_0(x)$
- let's allow one parton radiation (α_S order)
  dominant contribution from collinear radiation at small parton virtualities $t \ll Q^2$
- proton described by **dressed** PDF $q(x, M^2)$
  number of partons with fractional momentum x and virtuality up to M^2

$$q(x,M^2)\mathrm{d}x = q_0(x)\mathrm{d}x + \int_x^1 \mathrm{d}y q_0(y) P_{qq}\left(\frac{x}{y}\right)\frac{\mathrm{d}x}{y}\int_\beta^{M^2}\frac{\alpha_s(t)}{2\pi}\frac{\mathrm{d}t}{t} + \int_x^1 \mathrm{d}y g_0(y) P_{qg}\left(\frac{x}{y}\right)\frac{\mathrm{d}x}{y}\int_\beta^{M^2}\frac{\alpha_s(t)}{2\pi}\frac{\mathrm{d}t}{t} + \mathcal{O}(\alpha_S^2)$$

- we can't send the gluon mass (β) to zero, but we can get rid of β by derivation with respect to M^2

$$\frac{\mathrm{d}q(x,M^2)}{\mathrm{d}M^2} = \frac{\alpha_s(M^2)}{2\pi}\frac{1}{M^2}\int_x^1\frac{\mathrm{d}y}{y}\left[P_{qq}\left(\frac{x}{y}\right)q_0(y) + P_{qg}\left(\frac{x}{y}\right)g_0(y)\right] + \mathcal{O}(\alpha_S^2)$$

- $q_0(x)$ and $q(x,M^2)$ differ by quantity proportional to α_S; $M^2/\mathrm{d}M^2 = 1/\mathrm{d}\ln M^2$

$$\frac{\mathrm{d}q(x,M^2)}{\mathrm{d}\ln M^2} = \frac{\alpha_s(M^2)}{2\pi}\int_x^1\frac{\mathrm{d}y}{y}\left[P_{qq}\left(\frac{x}{y}\right)q(y,M^2) + P_{qg}\left(\frac{x}{y}\right)g(y,M^2)\right] + \mathcal{O}(\alpha_S^2)$$

DGLAP evolution equations

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi

$$\frac{\mathrm{d}q(x,M^2)}{\mathrm{d}\ln M^2} = \frac{\alpha_s(M^2)}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\left[P_{qq}\left(\frac{x}{y}\right)q(y,M^2) + P_{qg}\left(\frac{x}{y}\right)g(y,M^2)\right]$$

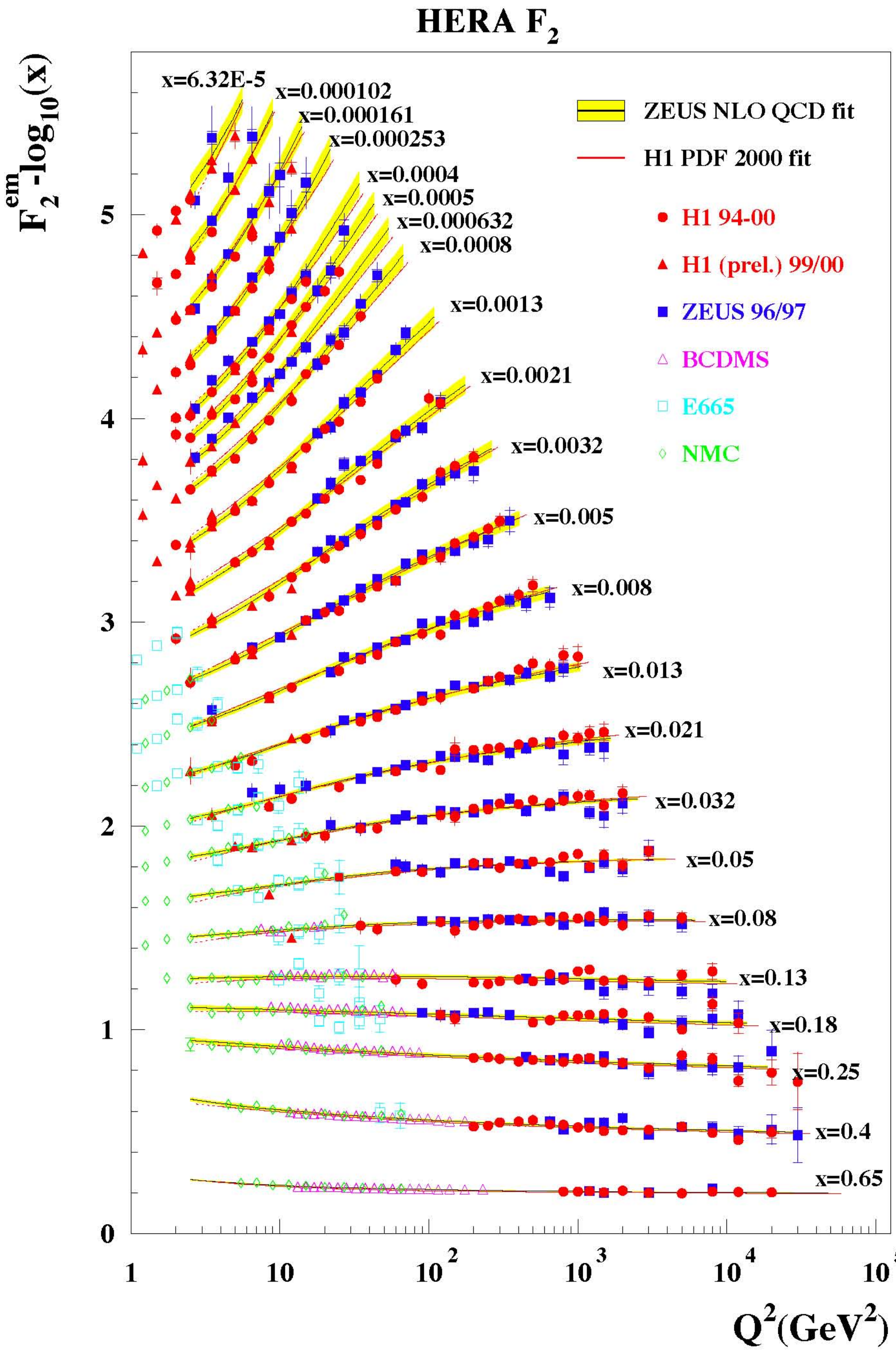
$$\frac{\mathrm{d}g(x,M^2)}{\mathrm{d}\ln M^2} = \frac{\alpha_s(M^2)}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\left[P_{gq}\left(\frac{x}{y}\right)\sum_q^{\text{flav.}}\left\{q(y,M^2)+\bar{q}(y,M^2)\right\} + P_{gg}\left(\frac{x}{y}\right)g(y,M^2)\right]$$

- integro-differential equations that governs the evolution of PDF on parton virtualities
- derived for just one parton emission – in next lecture, we will see that they hold in a dominant kinematic region for multiple emissions
  - NLO splitting functions: Curci, Furmanski, and Petronzio '80
  - NNLO splitting functions: Mochm Vernaseren, Vogt '04
- remarkable achievement of QCD - description of dependence of $F_2(x,Q^2)$ on Q^2

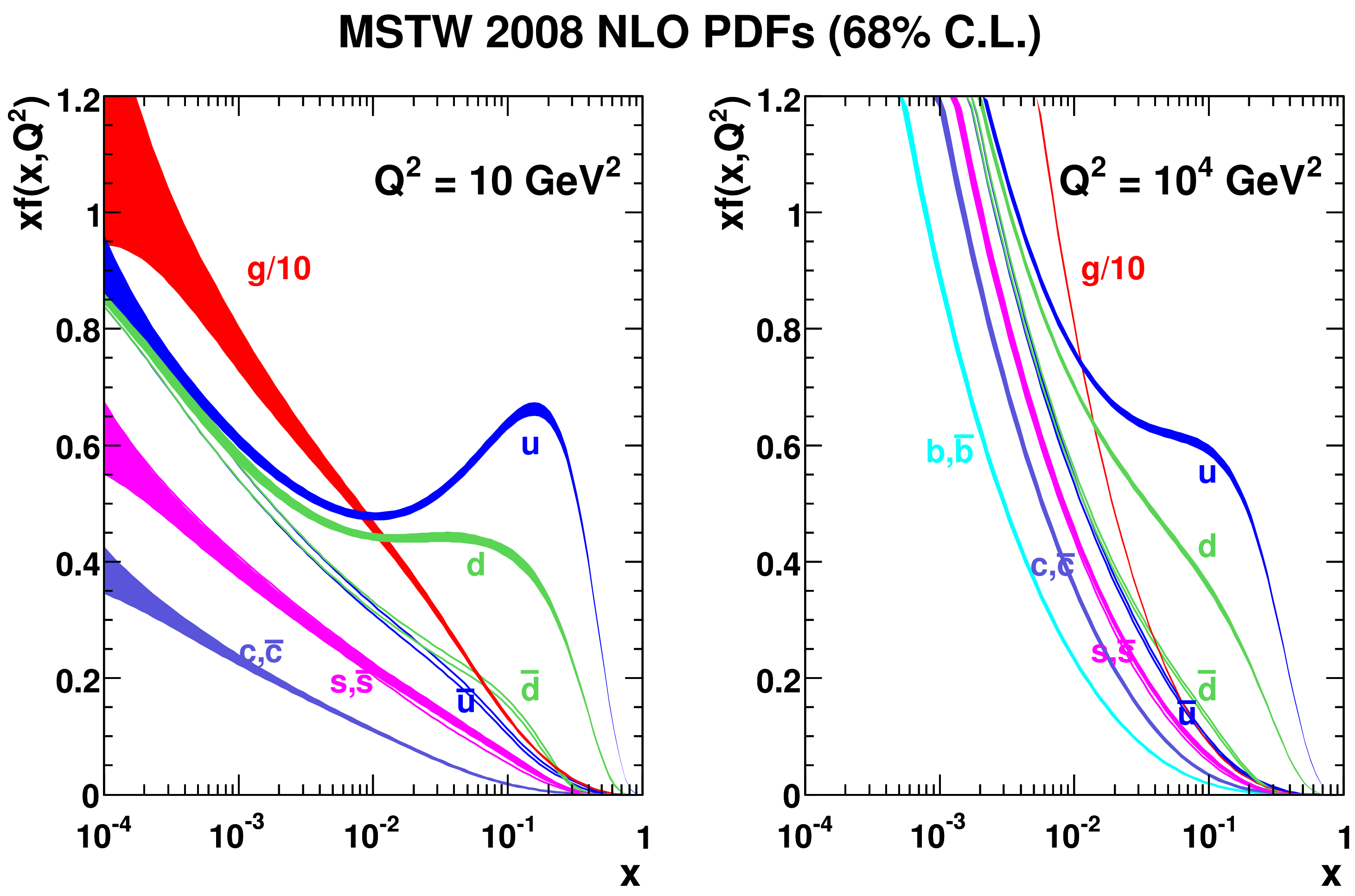
$$F_2(x,Q^2) = x\sum_q^{\text{flav.}} e_q^2\left[q(x,Q^2)+\bar{q}(x,Q^2)\right]$$

HERA

- $e^-/e^+ + p$ collider
- $E_{cms} = 318$ GeV
  - 27.5 GeV electrons
  - 920 GeV protons



PDF global fits



- PDF parameterization at low scale Q_0^2

$$q(x,Q_0^2) = A_q x^{B_q}(1-x)^{C_q}(1+D_q x+\ldots)$$
$$g(x,Q_0^2) = A_g x^{B_g}(1-x)^{C_g}(1+D_g x+\ldots)$$

- QCD
  - $q(x,Q_0^2) \;\rightarrow\; q(x,Q^2_{\text{exp}})$
  - theory prediction $T_i(\alpha_S, \text{PDF pars})$
- Global fit

$$\chi^2 = \sum_i \frac{(D_i - T_i)^2}{\sigma_i^2}$$

For reading and for exercises

- For reading (J. Chyla textbook)
  - chapter 9.1 (Mass singularities in perturbation theory)
  - chapter 9.2 (KLN theorem)
- Exercise
  - exercise 7.2 (Process $e^+e^- \to q\bar{q}g$ with finite gluon mass)
  - exercise 8.1 (Splitting functions in QCD)