Quarks, Partons, and Quantum Chromodynamics

Lecture 10

Tests of QCD in electron-positron collisions

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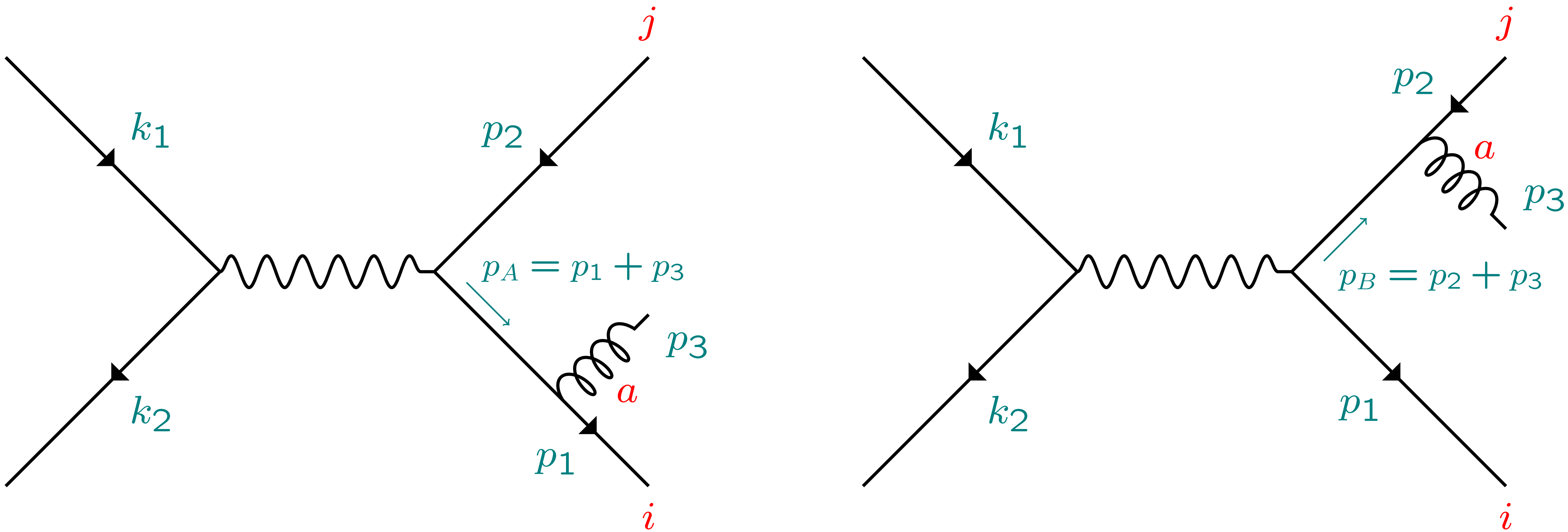
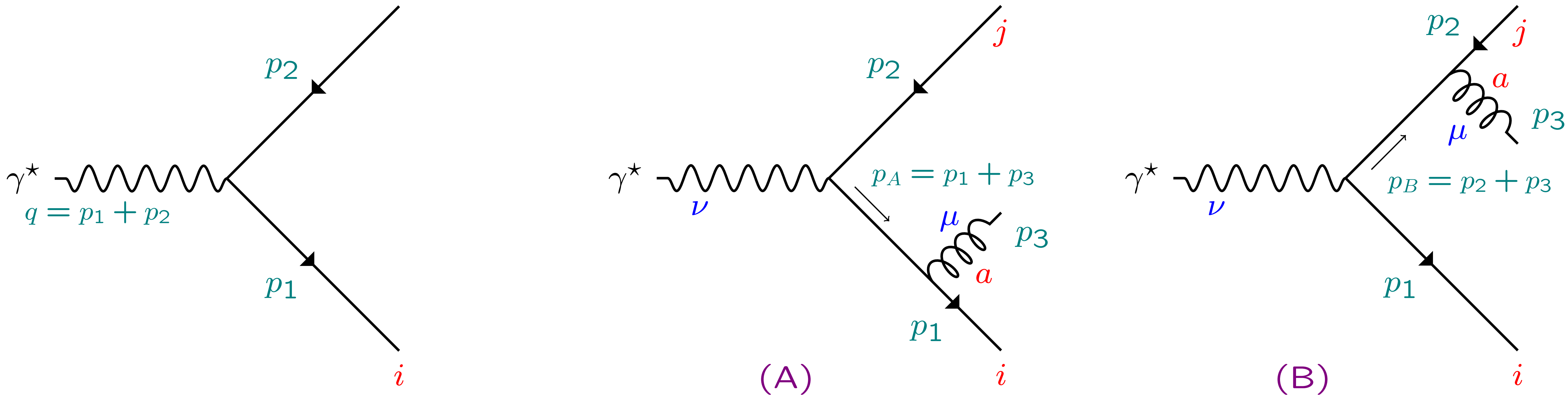
Process $e^+e^- \to q\bar{q}g$



- Born approximation (tree level)
- neglect electron and quark masses
- only QED (neglect contribution from Z)

$$x_i = \frac{2E_i}{\sqrt{s}}, \qquad i = 1\ (q),\ i = 2\ (\bar{q}),\ i = 3\ (g) \qquad \sqrt{s} = Q$$

$e^+e^- \to q\bar{q}g$ and $\gamma^\star \to q\bar{q}g$



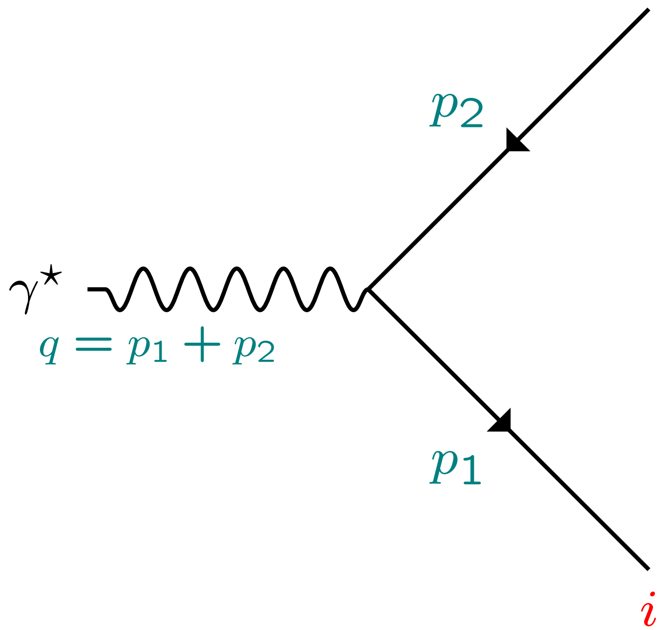
- trick: $\sigma_{ee\to qq}/\sigma_{ee\to qqg} = \Gamma_{\gamma^\star\to qq}/\Gamma_{\gamma^\star\to qqg}$ R. Field, Applications of Perturbative QCD
- formula for n body decay at particle rest frame

$$\mathrm{d}\Gamma = \frac{1}{2E_{\mathrm{CMS}}} |\mathcal{M}|^2 \, \mathrm{d}^{3n} R_n, \qquad \text{with} \qquad \mathrm{d}^{3n} R_n = (2\pi)^4 \prod_i^n \frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i} \delta^{(4)}(q - \sum_i^n p_i)$$

- more simple, due to lack of incoming electron tensor $L_{\mu\nu}$
- sum over all 4 photon and gluon polarization states gives $\sum_\lambda \varepsilon^\star_\mu(\lambda)\varepsilon_\nu(\lambda) = -g_{\mu\nu}$

  Ward identity ($\varepsilon_\mu(k)\mathcal{M}^\mu \;\Rightarrow\; k_\mu\mathcal{M}^\mu = 0$) ensures we can apply it on external photons/gluons as well

$\gamma^\star \to q\bar{q}$



Matrix element is given by the contraction of charge current

$$J^\mu = \bar{u}(p_1, s_1)(-iee_q)\gamma^\mu v(p_2, s_2)$$

with photon polarization vector $\varepsilon_\mu(\lambda)$. The square of the matrix element is then given by the contraction of corresponding quark tensor $L^{\mu\nu} = J^\mu J^{\dagger\nu}$ and photon tensor $L^\gamma_{\mu\nu} = \varepsilon_\mu(\lambda)\varepsilon^*_\mu(\lambda)$. Summing over all polarization states of quarks and antiquarks (s_1 and s_2), and over photon polarizations λ we have

$$\sum_{s_1,s_2,\lambda} |\mathcal{M}|^2 = \sum_{s_1,s_2,\lambda} L^{\mu\nu}\varepsilon_\mu(\lambda)\varepsilon^*_\nu(\lambda) = -\sum_{s_1,s_2} L^{\mu\nu}g_{\mu\nu} = -\sum_{s_1,s_2} L^\mu_\mu.$$

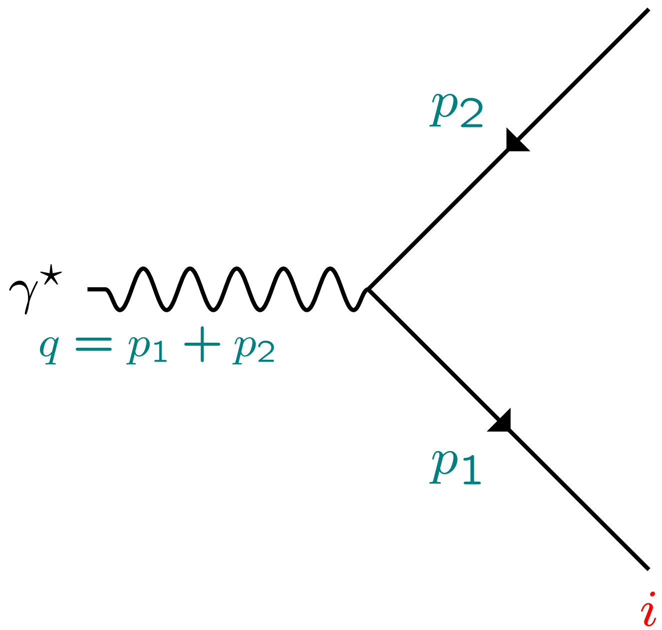
For the massless case, the matrix element summed over spins is

$$\sum_{s_1,s_2,\lambda} |\mathcal{M}|^2 = -\sum_{s_1,s_2} L^\mu_\mu = -J^{\dagger\mu}J_\mu = -e^2e_q^2\mathrm{Tr}(\not{p}_1 \underbrace{\gamma^\mu \not{p}_2 \gamma_\mu}_{-2\not{p}_2}) = 2e^2e_q^2\mathrm{Tr}(\not{p}_1\not{p}_2) = 8e^2e_q^2 \overbrace{(p_1 \cdot p_2)}^{s=(p_1+p_2)^2=2p_1\cdot p_2} = 4e^2e_q^2 s.$$

The decay product phase space d^6R_2 integrated over the delta function is

$$\mathrm{d}^6R_2 \ \to \ \mathrm{d}^2R_2 = (2\pi)^4 \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\delta^{(4)}(q - p_1 - p_2) = \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^3\vec{p}_1}{2E_1}\frac{1}{2E_2}\delta^{(1)}\overbrace{(\sqrt{s} - E_1 - E_2)}^{\text{in CMS, } E_1=E_2}$$

$$\overset{\text{sph. coor.}}{=} \frac{1}{16\pi^2}\int \frac{E_1^2\mathrm{d}E_1\mathrm{d}\Omega}{E_1^2} \underbrace{\frac{1}{2}\,\delta(\sqrt{s}/2 - E_1)}_{\delta(g(x))=\delta(x-\bar{x})/|g'(\bar{x})|,\ g(\bar{x})=0} = \frac{1}{32\pi^2}\mathrm{d}\Omega.$$



We did not average over the initial spin states, we just need to remember to do the same when we will evaluate $q\bar{q}g$ final state. There is, however, an additional factor of 3 connected with the fact that quark can occur in three color states.

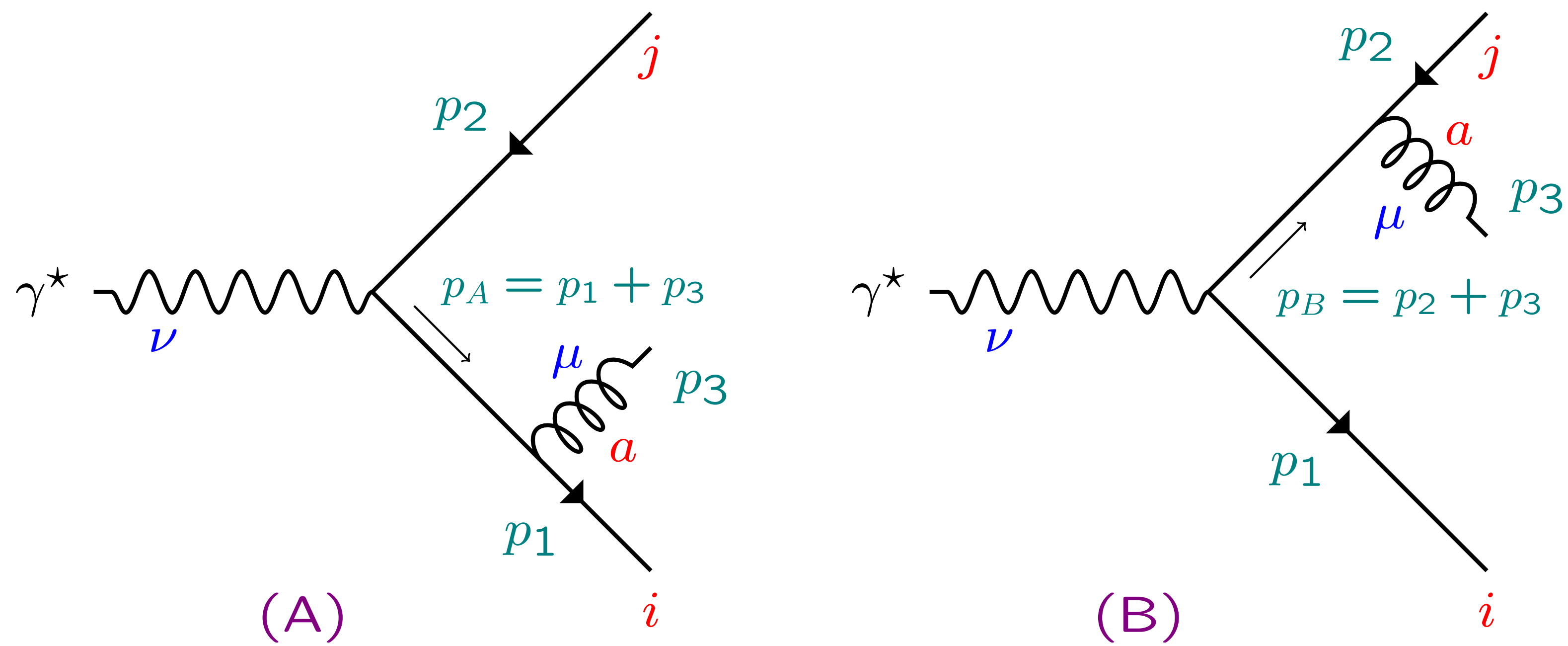
Also note that matrix element does not depend on scattering angle. Therefore the total decay width of $\gamma^\star \to q\bar{q}$ process is simply

$$\Gamma_0 = 3\,\frac{1}{2\sqrt{s}}\,4e^2e_q^2 s\,\frac{1}{32\pi^2}\int \mathrm{d}\Omega = \frac{1}{2\sqrt{s}}\,4e^2e_q^2 s\,\frac{1}{32\pi^2}\,4\pi = 3\,\frac{e^2}{4\pi}e_q^2\sqrt{s} = 3\,\alpha e_q^2\sqrt{s}.$$

Corresponding LO cross section for $e^+e^- \to q\bar{q}$ scattering is

$$\sigma_0 = 3\,\frac{4\pi\alpha^2}{3s}\sum_q^{flav.} e_q^2$$

$\gamma^\star \to q\bar{q}g$



The two leading order Feynman diagrams are shown above. The corresponding matrix elementsreads

$$i\mathcal{M}_A = \bar{u}(p_1)(-ig\gamma^\mu T^a_{ij})\frac{i\not{p}_A}{p_A^2}(-iee_q\gamma^\nu)v(p_2)\varepsilon^\star_{g\mu}(p_3)\varepsilon_\nu(q),$$

$$i\mathcal{M}_B = \bar{u}(p_1)(-iee_q\gamma^\nu)\frac{-i\not{p}_B}{p_B^2}(-ig\gamma^\mu T^a_{ij})v(p_2)\varepsilon^\star_{g\mu}(p_3)\varepsilon_\nu(q).$$

Note the minus sign in front of p_B, it is connected with the direction of 4-momentum p_B in the propagator of anti-quark. Square of the matrix element equals to

$$|\mathcal{M}|^2 = (\mathcal{M}_A + \mathcal{M}_B)(\mathcal{M}_A^* + \mathcal{M}_B^*) = |\mathcal{M}_A|^2 + 2\text{Re}(\mathcal{M}_A\mathcal{M}_B^\star) + |\mathcal{M}_B|^2.$$

$\gamma^\star \to q\bar{q}g$

$$\sum |\mathcal{M}_A|^2 = T^a_{ij}T^a_{ji}\frac{g^2e^2e_q^2}{p_A^4}\mathrm{Tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu\not{p}_2\gamma^\sigma\not{p}_A\gamma^\tau)\underbrace{\varepsilon^\star_{g\mu}(p_3)\varepsilon_{g\tau}(p_3)}_{-g_{\mu\tau}}\underbrace{\varepsilon^\star_\nu(q)\varepsilon^\star_\sigma(q)}_{-g_{\nu\sigma}} = \underbrace{\mathrm{Tr}(T^aT^a)}_{\frac{1}{2}\delta_{aa}}\frac{g^2e^2e_q^2}{p_A^4}\mathrm{Tr}(\underbrace{\gamma_\mu\not{p}_1\gamma^\mu}_{-2\not{p}_1}\not{p}_A\underbrace{\gamma^\nu\not{p}_2\gamma_\nu}_{-2\not{p}_2}\not{p}_A)$$

$$= 4\frac{4g^2e^2e_q^2}{p_A^4}\mathrm{Tr}(\not{p}_1\not{p}_A\not{p}_2\not{p}_A) = 4\frac{4g^2e^2e_q^2}{p_A^4}4\left[2(p_1\cdot p_A)(p_2\cdot p_A) - p_A^2(p_1\cdot p_2)\right]$$

We will evaluate the scalar product terms in the CMS where the photon 4-momentum is $q = p_1 + p_2 + p_3 = (\sqrt{s}, 0, 0, 0)$. Introducing fractional energies $x_i = 2E_i/\sqrt{s}$, we can get scalar products of any momentum pair from

$$2\,p_1\cdot p_2 = (p_1+p_2)^2 = (q-p_3)^2 = q^2 - 2(q\cdot p_3) = s - 2\sqrt{s}E_3 = s(1-x_3).$$

In analogy, $2\,p_1\cdot p_3 = s(1-x_2)$ and $2\,p_2\cdot p_3 = s(1-x_1)$. For the scalar products with $p_A = p_1 + p_3$ term, we have

$$\begin{aligned}
2\,p_1\cdot p_A &= 2\,p_1\cdot p_1 + 2p_1\cdot p_3 = s(1-x_2),\\
2\,p_2\cdot p_A &= 2\,p_1\cdot p_2 + 2p_3\cdot p_2 = s(2-x_1-x_3) = sx_2,\\
p_A^2 &= (p_1+p_3)^2 = 2\,p_1\cdot p_3 = s(1-x_2),
\end{aligned}$$

where we used energy conservation, which expressed in fractional energies x_i reads $x_1 + x_2 + x_3 = 2$. The square of the matrix element is then

$$\sum |\mathcal{M}_A|^2 = 4\frac{8g^2e^2e_q^2}{p_A^4}\left[2(p_1\cdot p_A)2(p_2\cdot p_A) - p_A^2 2(p_1\cdot p_2)\right] = 4\frac{8g^2e^2e_q^2}{(1-x_2)^2}\left[(1-x_2)x_2 - (1-x_2)(1-x_3)\right]$$

$$= 4\frac{8g^2e^2e_q^2}{1-x_2}(x_2 + x_3 + x_1 - x_1 - 1) = 4\cdot 8g^2e^2e_q^2\frac{1-x_1}{1-x_2}.$$

$\gamma^\star \to q\bar{q}g$

In the case of square of matrix element $\mathcal{M}_B$, we can guess the result directly. With respect to diagram (A), we only need to exchange quark for anti-quark, $p_1 \leftrightarrow p_2$ $(p_A \to p_B)$

$$\sum |\mathcal{M}_B|^2 = 4 \cdot 8g^2e^2e_q^2\frac{1-x_2}{1-x_1}.$$

The interference term is the most difficult to evaluate. Here we have

$$\sum 2\mathcal{M}_A\mathcal{M}_B^\star = 2T^a_{ij}T^a_{ji}\frac{g^2e^2e_q^2}{p_A^2p_B^2}\mathrm{Tr}[\not{p}_1\gamma^\mu(i\not{p}_A)\gamma^\nu\not{p}_2\gamma_\mu(-i)^\star\not{p}_B\gamma_\nu] = -4\frac{2g^2e^2e_q^2}{p_A^2p_B^2}\mathrm{Tr}[\not{p}_1\underbrace{\gamma^\mu\not{p}_A\gamma^\nu\not{p}_2\gamma_\mu}_{-2\not{p}_2\gamma^\nu\not{p}_A}\not{p}_B\gamma_\nu]$$

$$= 4\frac{4g^2e^2e_q^2}{p_A^2p_B^2}\mathrm{Tr}(\not{p}_1\not{p}_2\underbrace{\gamma^\nu\not{p}_A\not{p}_B\gamma_\nu}_{4p_A\cdot p_B}) = 4\frac{16g^2e^2e_q^2(p_A\cdot p_B)}{p_A^2p_B^2}\underbrace{\mathrm{Tr}(\not{p}_1\not{p}_2)}_{4p_1\cdot p_2} = 4\cdot 16g^2e^2e_q^2\frac{4(p_A\cdot p_B)(p_1\cdot p_2)}{p_A^2p_B^2}.$$

We will need the following additional scalar products, $p_B^2 = s(1-x_1)$ and

$$2p_A\cdot p_B = 2(p_1+p_3)\cdot(p_2+p_3) = 2p_1\cdot p_2 + 2p_1\cdot p_3 + 2p_2\cdot p_3 = s(1-x_3+1-x_2+1-x_1) = s.$$

The interference term then simplifies to

$$\sum 2\mathcal{M}_A\mathcal{M}_B^\star = 4\cdot 16g^2e^2e_q^2\frac{1-x_3}{(1-x_2)(1-x_1)}.$$

For the sum of the square of the whole matrix element we have then

$$\sum |\mathcal{M}|^2 = 4 \cdot 8g^2e^2e_q^2 \left[\frac{1-x_1}{1-x_2} + \frac{1-x_2}{1-x_1} + \frac{2(1-x_3)}{(1-x_2)(1-x_1)}\right] = 4 \cdot 8g^2e^2e_q^2 \frac{(1-x_1)^2 + (1-x_2)^2 + 2 - 2x_3}{(1-x_2)(1-x_1)}$$
$$= 4 \cdot 8g^2e^2e_q^2 \frac{x_1^2 + x_2^2 + 4 - 2x_1 - 2x_2 - 2x_3}{(1-x_2)(1-x_1)} = 4 \cdot 8g^2e^2e_q^2 \frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)}.$$

Evaluation 3-body decay phase space

$$\mathrm{d}^9 R_3 = (2\pi)^4 \prod_i^3 \frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i} \delta^{(4)}(q - \sum_i^3 p_i),$$

As the matrix element depends only on two kinematic variables, we would like to integrate over the remaining degrees of freedom. We can integrate over one particle momentum ($\vec{p_3}$), eating the three spatial components of delta function

$$\mathrm{d}^9 R_3 \to \mathrm{d}^6 R_3 = \frac{1}{(2\pi)^5 2E_3} \frac{\mathrm{d}^3\vec{p}_1}{2E_1} \frac{\mathrm{d}^3\vec{p}_2}{2E_2} \delta(\sqrt{s} - E_1 - E_2 - E_3) = \frac{1}{(2\pi)^5 2E_3} \frac{\mathrm{d}E_1 E_1^2}{2E_1} \mathrm{d}\Omega_1 \frac{\mathrm{d}E_2 E_2^2}{2E_2} \mathrm{d}\Omega_2\, \delta(\sqrt{s} - E_1 - E_2 - E_3).$$

We switched to spherical coordinates to integrate over the remaining delta function and other degrees of freedom. In particular, the matrix element depends only on quark and antiquark energies. We can therefore integrate over the direction of quark, i.e. over solid angle Ω_1.

$\gamma^\star \to q\bar{q}g$

We have to be more careful about the integration over the solid angle Ω_2. Once the direction of quark is fixed, the direction of antiquark is not arbitrary as the angle between quark and anti-quark together with their energies determines the energy of gluon E_3. Therefore in the case of antiquark, we select the z-axis of spherical coordinate system in the direction of quark and the polar angle is then the angle between quark and antiquark ϑ_{12}. From the momentum conservation we have at CMS

$$E_3^2 = |\vec{p}_3|^2 = (\vec{p}_1 + \vec{p}_2)^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\vartheta_{12}.$$

We will use the remaining delta function to integrate over angle ϑ_{12}. The function inside of the delta function is

$$g(\cos\vartheta_{12}) = \sqrt{s} - E_1 - E_2 - \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\vartheta_{12}}.$$

The first derivate in then

$$\frac{\partial g}{\partial\cos\vartheta_{12}} = \frac{1}{2}\frac{2E_1E_2}{\sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\vartheta_{12}}} = \frac{E_1E_2}{E_3}$$

The azimuthal angle ϕ_2 of antiquark in the coordinate system is free as the gluon energy does not depend on it, neither the matrix element. It can be integrated over as well, yielding a factor of 2π. At the end, we have just two independent quantities, quark and antiquark energies

$$\mathrm{d}^2R_3 = \frac{1}{(2\pi)^5 2E_3}\frac{1}{4}\mathrm{d}E_1E_1\mathrm{d}E_2E_2\underbrace{\int\mathrm{d}\Omega_1}_{4\pi}\underbrace{\int\mathrm{d}\phi_2}_{2\pi}\underbrace{\int\mathrm{d}(\cos\vartheta_{12})\delta(g(\cos\vartheta_{12}))}_{E_3/(E_1E_2)} = \frac{\mathrm{d}E_1\mathrm{d}E_2}{2^5\pi^3} = \frac{s}{2^7\pi^3}\mathrm{d}x_1\mathrm{d}x_2.$$

$\gamma^\star \to q\bar{q}g$

Now we have everything in hand to calculate the decay width,

$$\mathrm{d}\Gamma = \frac{1}{2\sqrt{s}}|\mathcal{M}|^2 \frac{s}{2^7\pi^3}\mathrm{d}x_1\mathrm{d}x_2 = \frac{\sqrt{s}}{2^8\pi^3}|\mathcal{M}|^2\mathrm{d}x_1\mathrm{d}x_2.$$

Plugging in the matrix element, and factoring out Γ_0 we arrive at

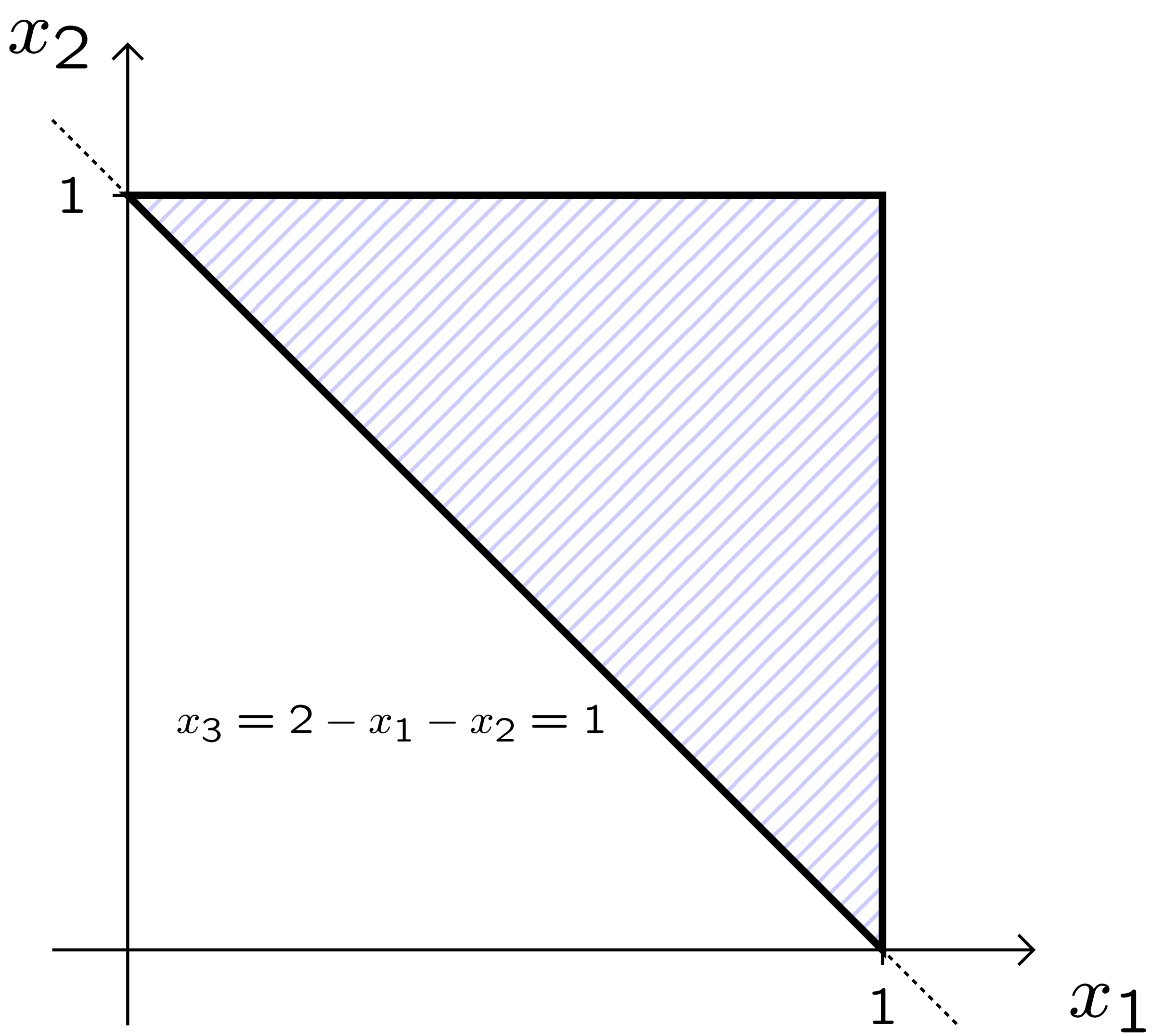
$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x_1\mathrm{d}x_2} = \frac{\sqrt{s}}{2^8\pi^3}4\cdot 8g^2e^2e_q^2\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)} = 4\frac{\alpha_S\alpha e_q^2}{2\pi}\sqrt{s}\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)}\frac{\Gamma_0}{3\alpha e_q^2\sqrt{s}} = \frac{4}{3}\frac{\alpha_S}{2\pi}\Gamma_0\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)}.$$

For the cross section we have then

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2} = \frac{4}{3}\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)} = C_F\frac{\alpha_S}{2\pi}\sigma_0\frac{x_1^2+x_2^2}{(1-x_2)(1-x_1)}}$$

In the last step, we noted that the color factor $4/3$ is the numerical factor C_F corresponding to the Casimir operator of SU(3) fundamental triplet.

Phase space of process $e^+e^- \to q\bar{q}g$



$$x_i \equiv \frac{2E_i}{\sqrt{s}}, \quad 1\text{ - quark, } 2\text{ - antiquark, } 3\text{ - gluon}$$

- energy conservation $(Q = \sqrt{s})$ $x_1 + x_2 + x_3 = 2$

$$\Rightarrow \; x_3 = 2 - x_1 - x_2, \qquad 0 < x_i < 2$$

- momentum conservation $(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0)$

$$\Rightarrow \; 0 < x_i < 1$$

- total cross section for process $e^+e^- \to q\bar{q}g$ is infinitely large

$$\sigma_{q\bar{q}g} = \int_0^1 \mathrm{d}x_1 \int_{1-x_1}^1 \mathrm{d}x_2 \; C_F \frac{\alpha_S}{2\pi} \sigma_0 \frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)} \to \infty$$

Event shapes in e^+e^-

Thrust T

$$T = \max_{|\vec{n}|=1} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

- the sum goes over all particles in the final state
- direction $\vec{n}$, which maximizes the ratio, is called **thrust axis**

- characterizes the shape of event
  - for pencil like event (all particles radiated along one direction $\vec{n}$) - $T = 1$
  - for spherically symmetric event - $T = \frac{1}{2}$

$$T = \frac{\int \mathrm{d}\Omega |\cos\vartheta|}{\int \mathrm{d}\Omega} = \frac{1}{4\pi} 2 \int_0^1 \mathrm{d}(\cos\vartheta) \cos\vartheta \int_0^{2\pi} \mathrm{d}\varphi = \frac{4\pi}{4\pi} \cdot \frac{1}{2} = \frac{1}{2}$$

- The LO term for $e^+e^- \to hadrons$ contains just quark and antiquark in the final state with their momenta being opposite in CMS. This is exactly a pencil-like event and the resulting thrust distribution is trivial, as $T = 1$.

Thrust distribution in e^+e^-

- The first non-trivial contribution to the thrust distribution comes from additional radiation of gluon ($e^+e^- \to q\bar{q}g$)
- The corresponding cross section depends only on two relevant kinematic variables, we picked x_1 and x_2
- to calculate the thrust distribution, we need to know the relation between thrust T and variables x_1 and x_2
- introducing fractional momenta $\vec{x}_i = \frac{2\vec{p}_i}{\sqrt{E}}$, thrust reads

$$T = \max_{|\vec{n}|=1} \frac{\sum_i |\vec{x}_i \cdot \vec{n}|}{\sum_i |\vec{x}_i|} = \frac{1}{2} \max_{|\vec{n}|=1} \sum_i |\vec{x}_i \cdot \vec{n}|,$$

where we used the energy conservation

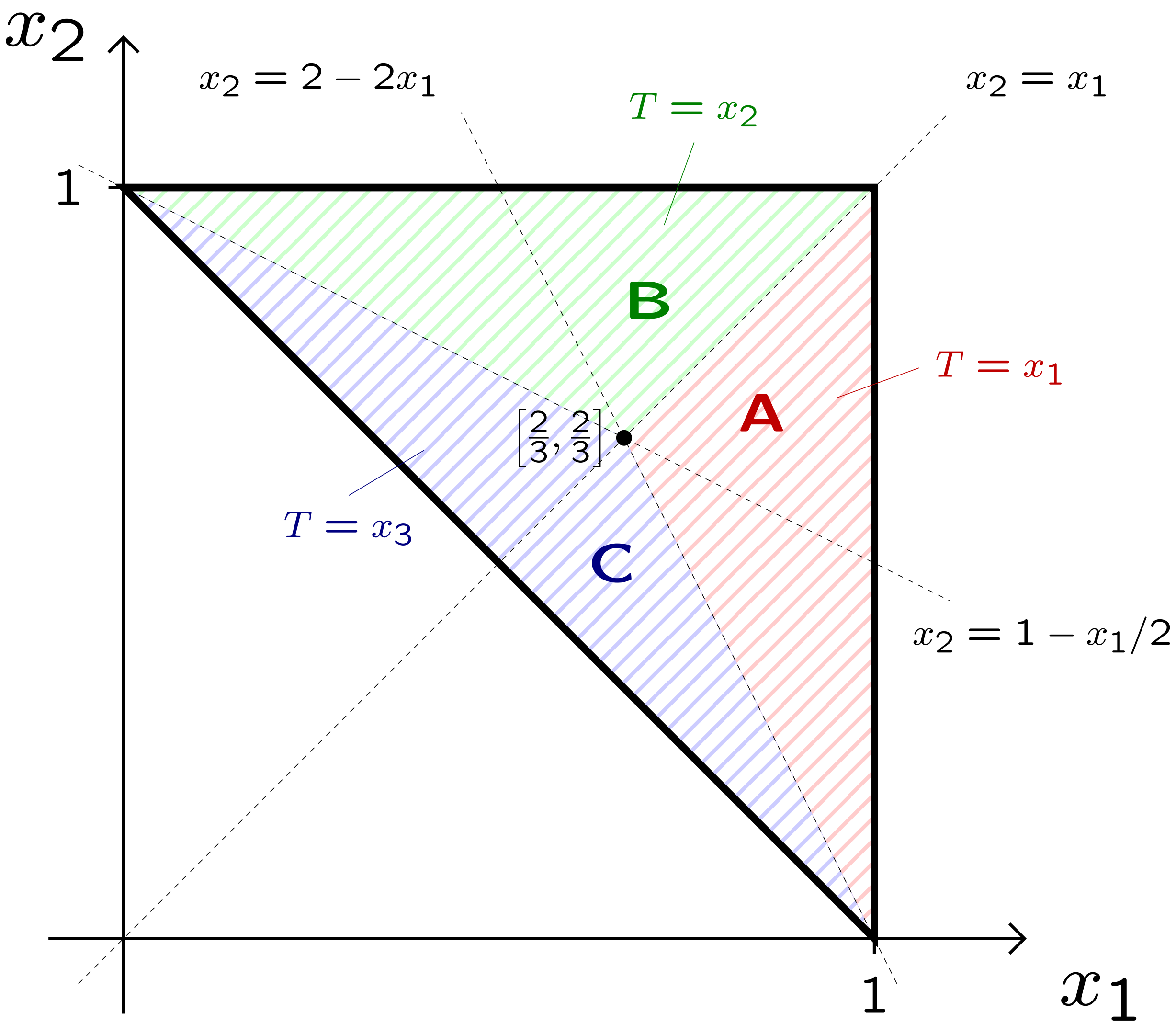
$$\sum_i |\vec{x}_i| = x_1 + x_2 + x_3 = 2.$$

- Let's pick a parton with largest energy and let it be a parton with index 1. Momentum conservation gives $\vec{x}_1 = -\vec{x}_2 - \vec{x}_3$.

$$\Rightarrow \quad |\vec{x}_1 \cdot \vec{n}| = |(\vec{x}_2 + \vec{x}_3) \cdot \vec{n}\,| \quad \Rightarrow \quad \sum_i |\vec{x}_i \cdot \vec{n}| = 2|\vec{x}_1 \cdot \vec{n}|$$

$$\Rightarrow \quad T(x_1, x_2) = \max\{x_1, x_2, x_3\} = \max\{x_1, x_2, 2 - x_1 - x_2\}$$

Thrust distribution in e^+e^-



$$T(x_1, x_2) = \max\{x_1, x_2, x_3\}$$

Region (A) - $\max\{x_1, x_2, x_3\} = x_1$

- $x_1 > x_2$
- $x_1 > x_3 = 2 - x_1 - x_2 \;\Rightarrow\; x_2 > 2 - 2x_1$

Region (B) - $\max\{x_1, x_2, x_3\} = x_2$

- exchange $x_1 \leftrightarrow x_2$

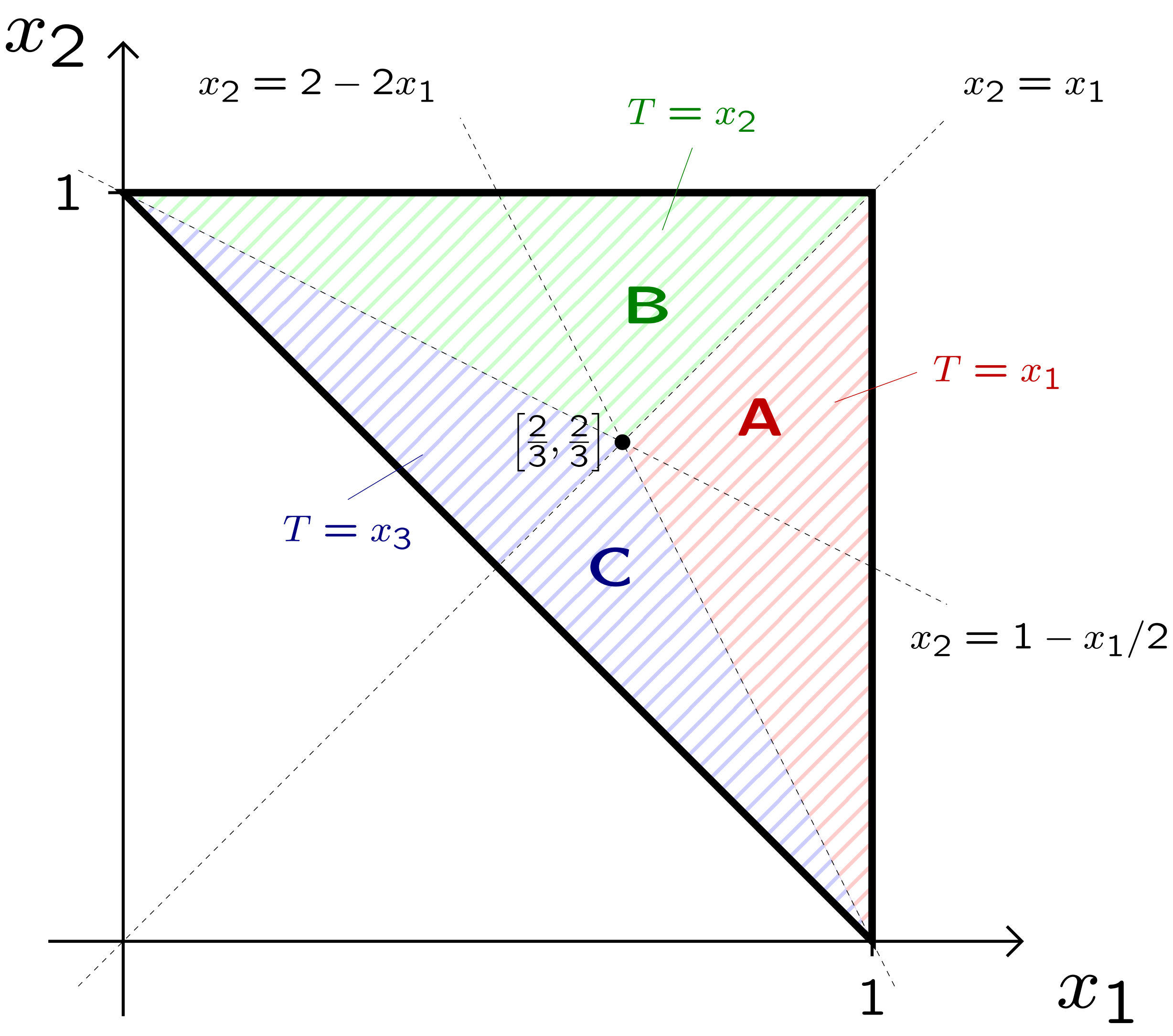
$\Rightarrow$ symmetric to (A) with respect to $x_1 = x_2$ axis

Region (C) - $\max\{x_1, x_2, x_3\} = x_3$

- the rest

$$\frac{\mathrm{d}\sigma}{\mathrm{d}T} = \int_0^1 \mathrm{d}x_1 \int_{x_1}^1 \mathrm{d}x_2 C_F \frac{\alpha_S}{2\pi}\sigma_0 \frac{x_1^2 + x_2^2}{(1 - x_2)(1 - x_1)}\delta(T - T(x_1, x_2))$$

Thrust distribution in e^+e^-



Region (B) - $\max\{x_1, x_2, x_3\} = x_2$

- from $x_1 \leftrightarrow x_2$ symmetry

$$I_B = I_A$$

Region (A) - $\max\{x_1, x_2, x_3\} = x_1$

$$I_A \equiv \frac{1}{C_F \frac{\alpha_S}{2\pi} \sigma_0} \frac{\mathrm{d}\sigma_A}{\mathrm{d}T} = \int_A \mathrm{d}x_1 \mathrm{d}x_2 \frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)} \delta(T - x_1)$$

We will use the delta function to integrate over $\mathrm{d}x_1$,

$$I_A = \int_{2-2T}^{T} \mathrm{d}x_2 \frac{T^2 + x_2^2}{(1-x_2)(1-T)}.$$

Remaining integral is straightforward to evaluate

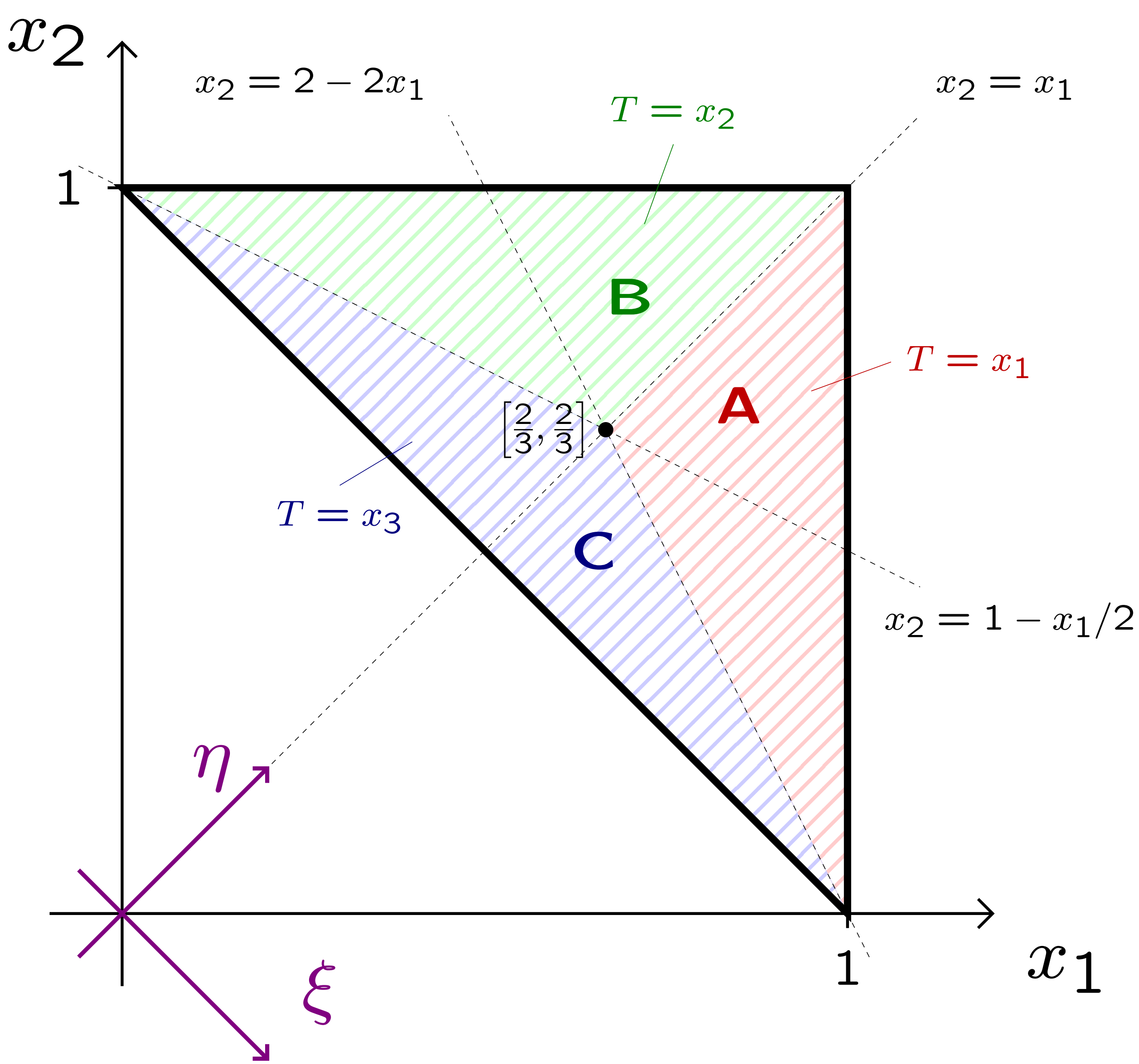
$$I_A = \frac{1}{1-T} \int_{2-2T}^{T} \mathrm{d}x_2 \frac{T^2 + x_2^2 - 1 + 1}{1 - x_2}$$

$$= -\frac{1}{1-T} \int_{2-2T}^{T} \mathrm{d}x_2 (1 + x_2) + \frac{1+T^2}{1-T} \int_{2-2T}^{T} \mathrm{d}x_2 \frac{1}{1-x_2}$$

$$\int_a^b \mathrm{d}x(1+x) = (b-a) + \frac{1}{2}(b^2 - a^2) = (b-a)\left[1 + \frac{1}{2}(b+a)\right]$$

$$I_A = -\frac{(3T-2)(2-T/2)}{1-T} + \frac{1+T^2}{1-T} \ln \frac{2T-1}{1-T}$$

Thrust distribution in e^+e^-



Region (C) - $\max\{x_1, x_2, x_3\} = x_3$

$$I_C = \int_C \mathrm{d}x_1 \mathrm{d}x_2 \frac{x_1^2 + x_2^2}{(1-x_2)(1-x_1)} \delta(T - 2 + x_1 + x_2)$$

To simplify our life, let's introduce a substitution, under which we could easily integrate out the delta function

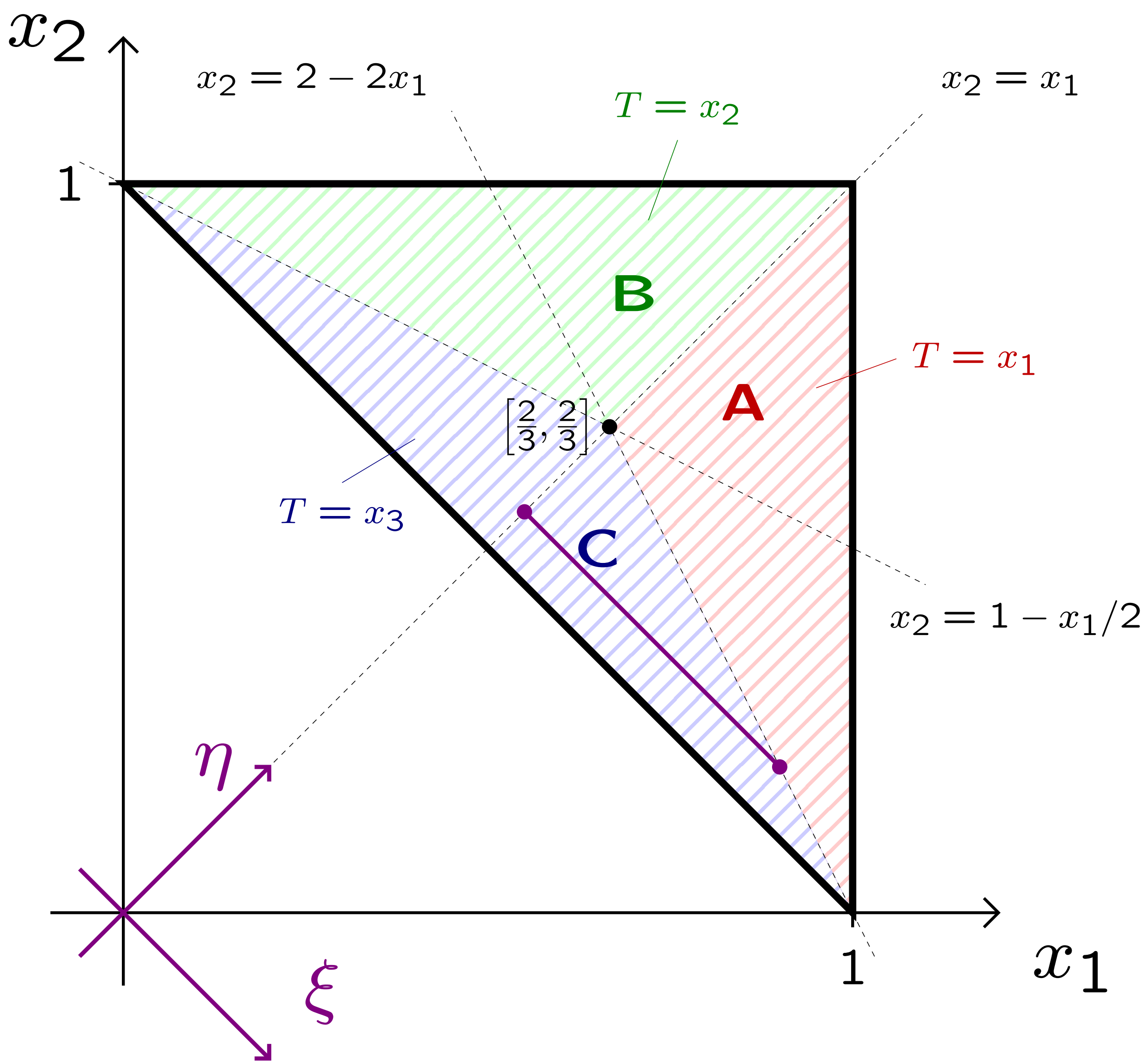
$$\xi = x_1 - x_2, \qquad \eta = x_1 + x_2,$$
$$x_1 = \frac{\xi + \eta}{2}, \qquad x_2 = \frac{-\xi + \eta}{2}.$$

This corresponds to a new coordinate system, with η axis being $x_2 = x_1$ line and ξ axis being perpendicular to it. The delta function fixes the value of η to $\eta = 2 - T$.

The integrand in new coordinates can be obtained from

$$x_1^2 + x_2^2 = \frac{1}{4}\left[(\xi + \eta)^2 + (-\xi + \eta)^2\right] = \frac{\xi^2 + \eta^2}{2} = \frac{\xi^2 + (2-T)^2}{2},$$
$$(1 - x_1)(1 - x_2) = \frac{1}{4}(2 - \xi - \eta)(2 + \xi - \eta) = \frac{1}{4}(T - \xi)(T + \xi).$$

Thrust distribution in e^+e^-



Region (C) - $\max\{x_1, x_2, x_3\} = x_3$

We just need to integrate over line in area C with fixed value of η. The upper boundary is given by line $x_2 = 2 - 2x_1$. For a fixed value of thrust (and η) we get

$$\eta = x_1 + x_2 = 2 - x_1 = 2 - T,$$

i.e. $x_1 = T$. This is expected as in this point we touch the boundary of region A. This implies that $x_2 = 2 - 2T$

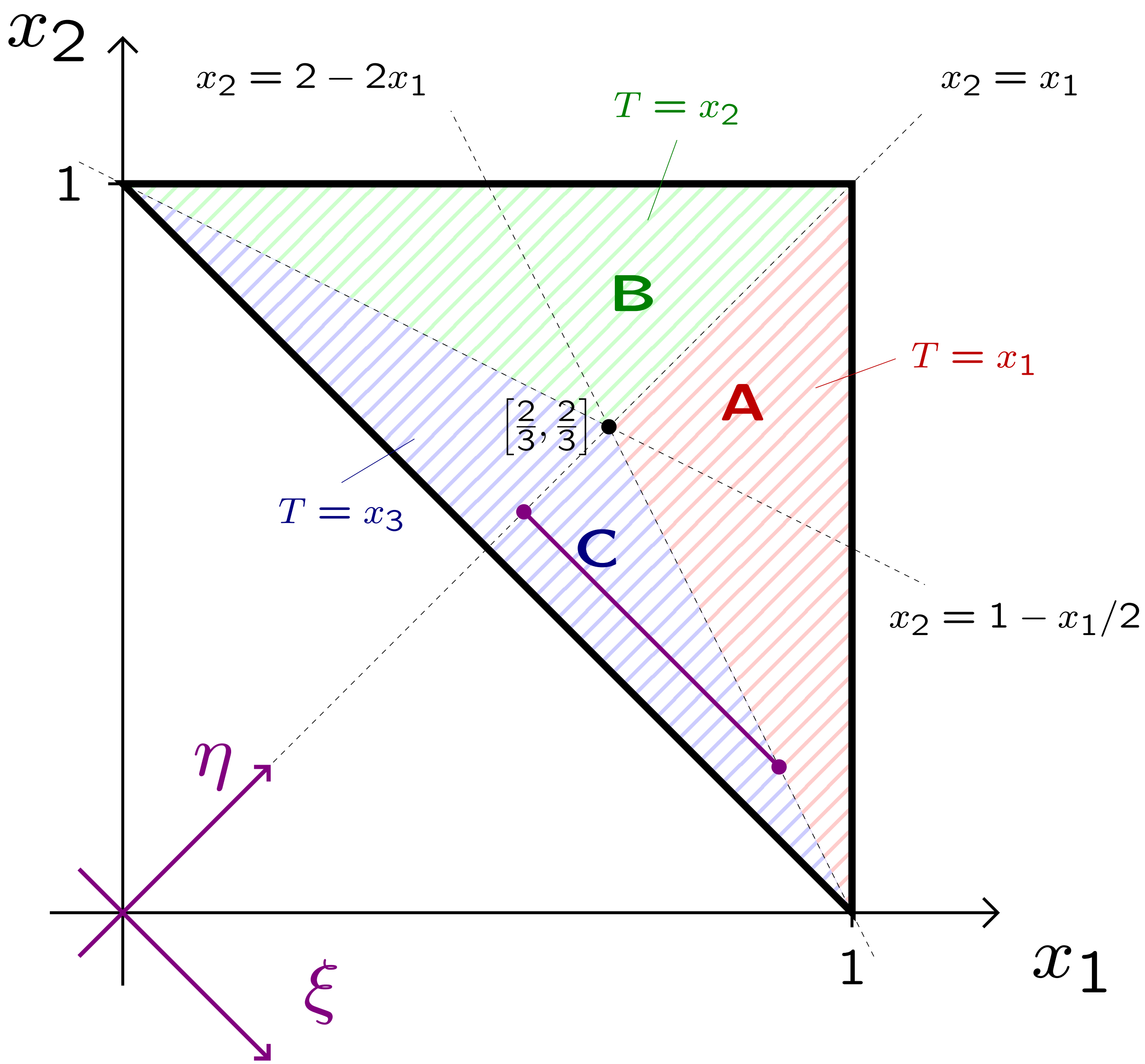
$$\Rightarrow \ \xi = x_1 - x_2 < T - 2 + 2T = 3T - 2.$$

We can just integrate over the half-plain $x_1 > x_2$, as the remaining part of region C must give, from the symmetry, the same contribution. In this case the lower boundary value of ξ is zero.

We still need the Jacobi determinant of the coordinate transformation

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_1}{\partial \eta} \\ \frac{\partial x_2}{\partial \xi} & \frac{\partial x_2}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Thrust distribution in e^+e^-



Region (C) - $\max\{x_1, x_2, x_3\} = x_3$

Putting all together we have

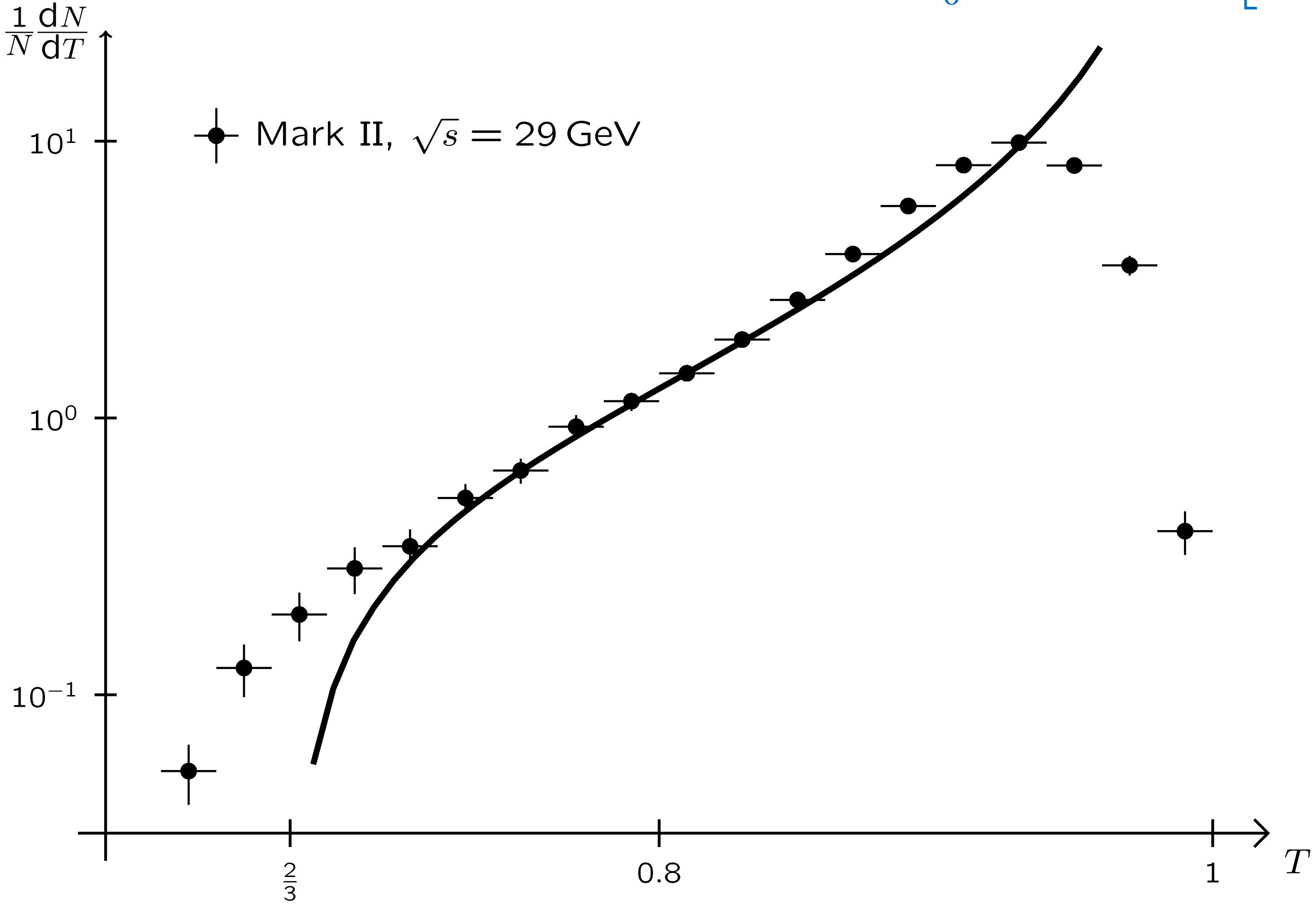
$$
\begin{aligned}
I_C &= 2\int_0^{3T-2} \mathrm{d}\xi \, \frac{1}{2}\frac{4}{2}\frac{\xi^2 + (2-T)^2}{(T-\xi)(T+\xi)} = 2\int_0^{3T-2} \mathrm{d}\xi \, \frac{\xi^2 - T^2 + T^2 + (2-T)^2}{(T-\xi)(T+\xi)} \\
&= -2(3T-2) + \frac{2[T^2 + (2-T)^2]}{2T}\int_0^{3T-2} \mathrm{d}\xi \left[\frac{1}{T-\xi} + \frac{1}{T+\xi}\right] \\
&= -2(3T-2) + \frac{4 - 4T + 2T^2}{T}\left[\ln\frac{T}{2-2T} + \ln\frac{4T-2}{T}\right] \\
&= 2\,\frac{2 - 2T + T^2}{T}\ln\frac{2T-1}{1-T} - 2(3T-2).
\end{aligned}
$$

$$I = I_A + I_B + I_C$$

$$
\begin{aligned}
\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}T} &= C_F\frac{\alpha_S}{2\pi}\left[2\frac{T + T^3 + 2 - 2T + T^2 - 2T + 2T^2 - T^3}{T(1-T)}\ln\frac{2T-1}{1-T} - (3T-2)\frac{4 - T + 2 - 2T}{1-T}\right] \\
&= C_F\frac{\alpha_S}{2\pi}\left[\frac{2(3T^2 - 3T + 2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]
\end{aligned}
$$

Thrust distribution in e^+e^-

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}T} = C_F\frac{\alpha_S}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]$$

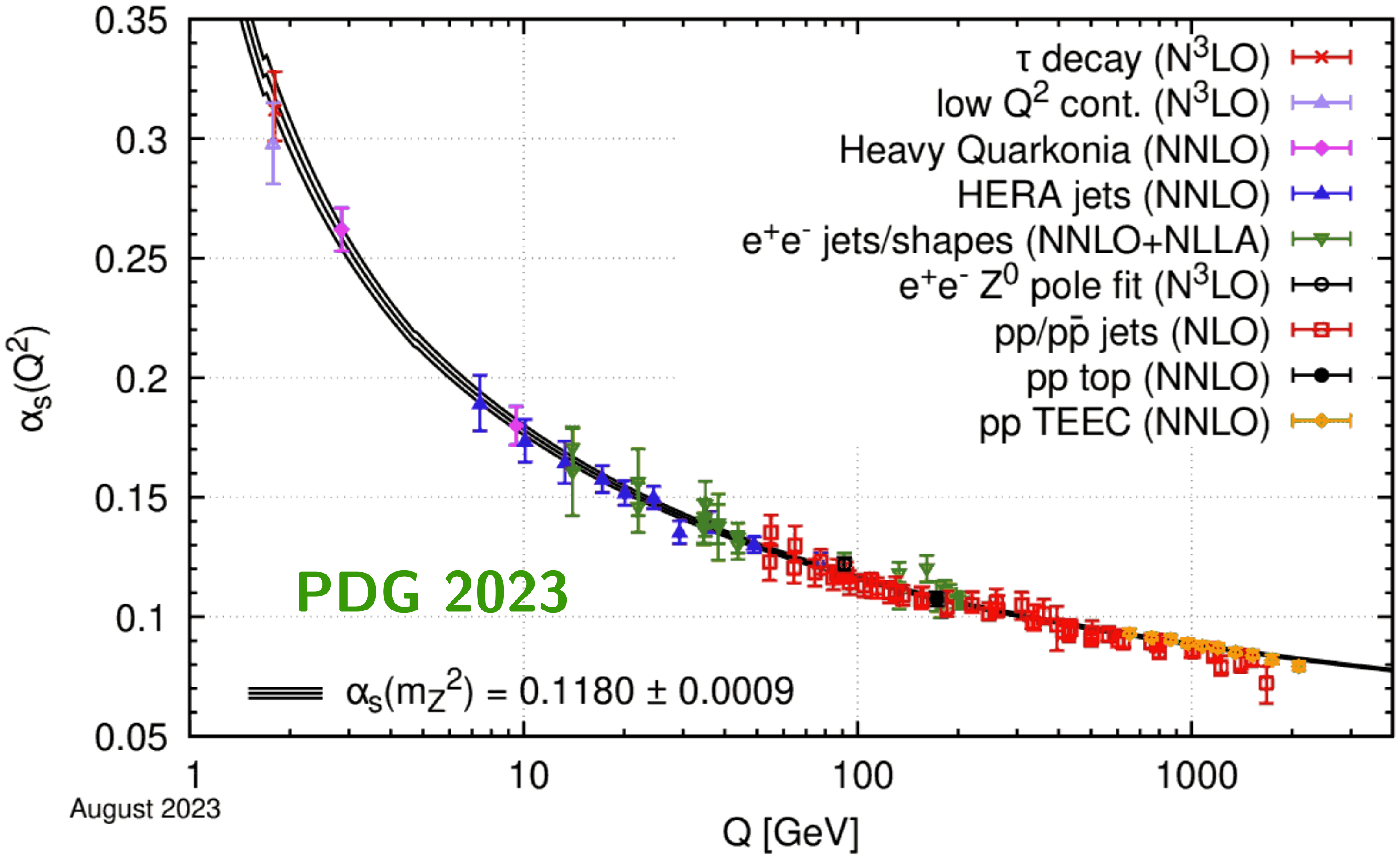


- Can you explain why the calculation fails in the regions where thrust approaches values of $2/3$ and 1?
- good agreement at middle ranges of thrust
- we can measure α_S there

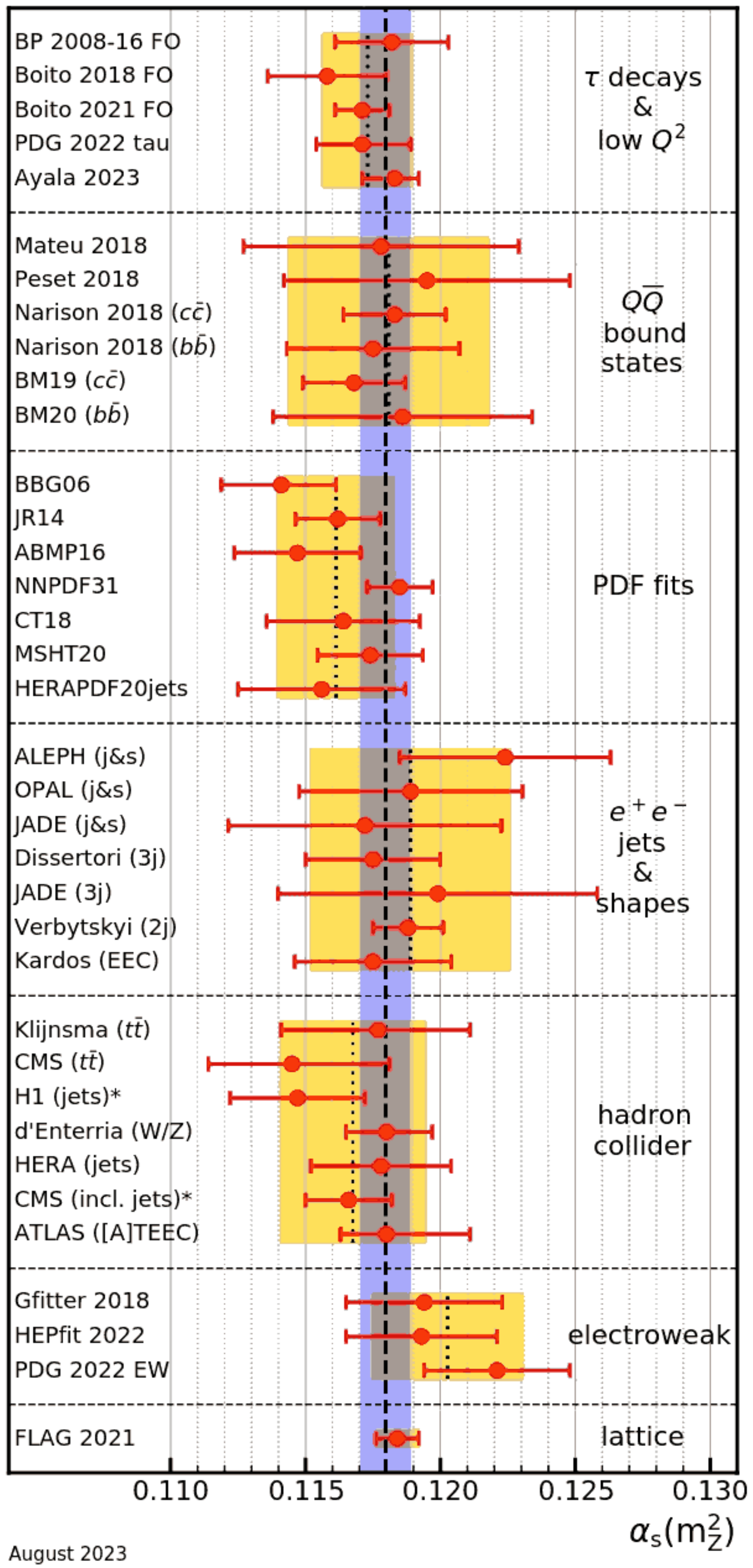
$$\frac{\alpha_S(30\text{ GeV})}{2\pi} \approx 0.07$$

- too rough - we need $ee \to q\bar{q}g$ at NLO to get running of α_S

Running of α_S in QCD



$\Rightarrow$



For reading and for exercises

- Exercise
  - exercise 7.1 (Process $e^+e^- \to q\bar{q}g$)
  - exercise 7.4 (Thrust distribution in e^+e^- collisions)