Quarks, Partons, and Quantum Chromodynamics

Lecture 9

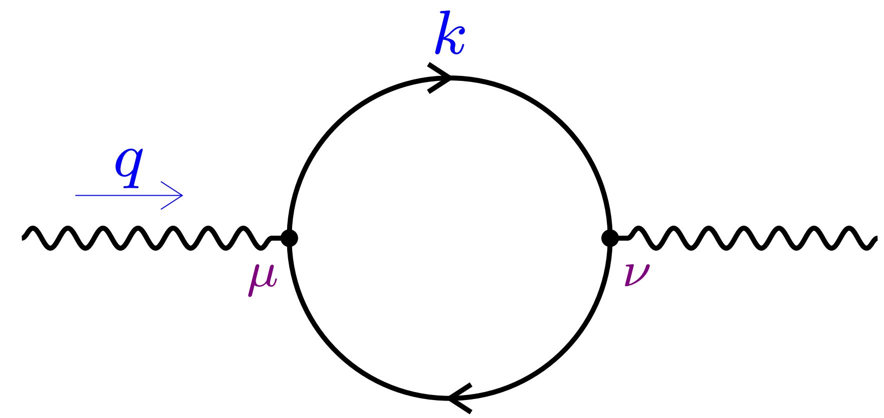
Running of strong coupling

Alexander Kupčo

http://www-hep2.fzu.cz/~kupco/QCD/

kupco@fzu.cz

Treatment of UV divergencies in QFT



- **regularization** - making the infinite loops finite

  cut-off (Λ), Pauli-Villars (M), dimensional regularization (ε)

- **renormalization**

  UV divergencies are absorbed by rescaling of Lagrangian components

  "Bare" quantities $\mathcal{L} = -\frac{1}{4} F_B^{\mu\nu} F_{B\mu\nu} + i\bar{\Psi}_B \not\partial \Psi_B - m_B \bar{\Psi}_B \Psi_B + e_B \bar{\Psi}_B \gamma^\mu \Psi_B A_{B\mu}$

  replaced by renormalized ones $\Psi_B \equiv Z_2^{1/2} \Psi_R, \quad A_B^\mu \equiv Z_3^{1/2} A_R^\mu, \quad m_B \equiv Z_m m_R, \quad e_B \equiv Z_e e_R,$

- interaction term Z_1: $e_B \bar{\Psi}_B \gamma^\mu \Psi_B A_{B\mu} \equiv Z_1 e_R \bar{\Psi}_R \gamma^\mu \Psi_R A_{R\mu} = \frac{Z_1}{Z_2 Z_3^{1/2}} e_R \bar{\Psi}_B \gamma^\mu \Psi_B A_{B\mu} \Rightarrow e_B = e_R \frac{Z_1}{Z_2 Z_3^{1/2}}$

$$\mathcal{L} = -\frac{1}{4} Z_3 F_R^{\mu\nu} F_{R\mu\nu} + i Z_2 \bar{\Psi}_R \not\partial \Psi_R - Z_m Z_2 m_R \bar{\Psi}_R \Psi_R + Z_1 e_R \bar{\Psi}_R \gamma^\mu \Psi_R A_{R\mu}$$

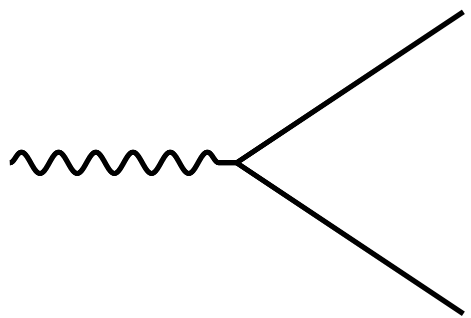
- introducing $Z_i \equiv 1 + \delta Z_i \quad \Rightarrow \quad \mathcal{L} = \mathcal{L}_{\text{orig}}^{\text{(in R)}} + (\delta Z)_{\text{counter terms}}$
- **renormalization scheme** - counter terms are chosen in such way to cancel divergencies of original $\mathcal{L}$
- **renormalizible QFT** - divergnecies can be absorbed in all orders of pQFT

  interpretation - physics at large scales M has no direct influence at low scales

Renormalization factors Z_i in NLO

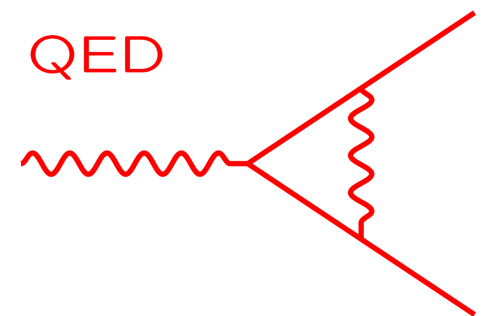
$$Z_i \quad = \quad 1 \quad (LO) \quad + \quad \delta Z_i \quad (NLO)$$

Z_1
vertex

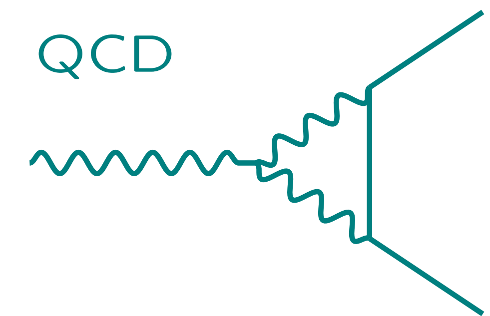


$+\ \delta Z_1$

QED

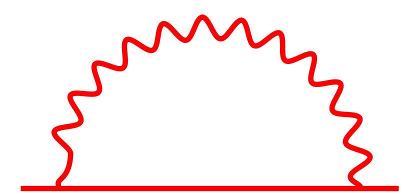


QCD



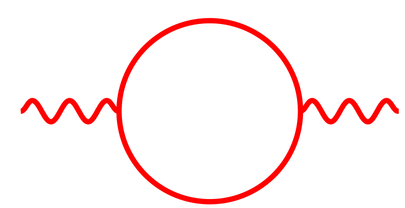
Z_2
fermion
propagator

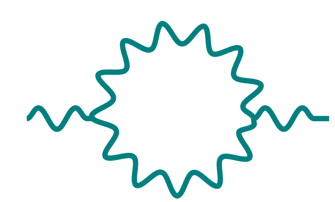
$+\ \delta Z_2$



Z_3
gauge field
propagator

$+\ \delta Z_3$

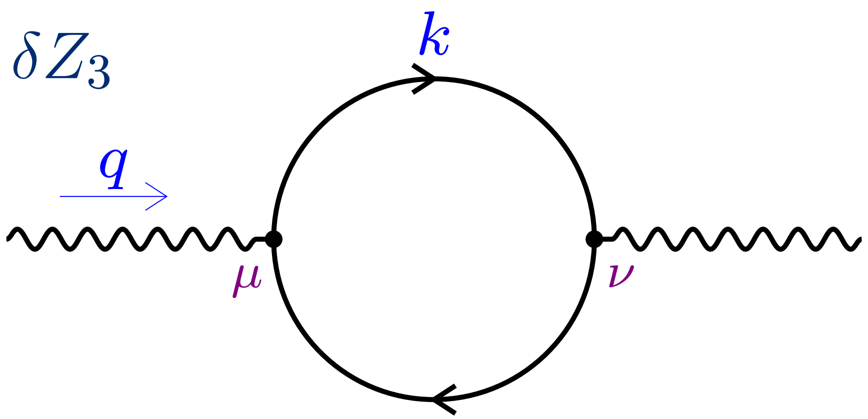




- in QED $Z_1 = Z_2$ (follows from Ward indentity) $\Rightarrow$ we need to compute just Z_3 to get running of α
- in QCD we need to calculate Z_1, Z_2, and Z_3 to get running of α_S

Running of α in QED

J. Chyla, chapter 8.2.1



- QED: $Z_1 = Z_2 \quad \Rightarrow \quad e_B = Z_3^{-1/2} e_R$

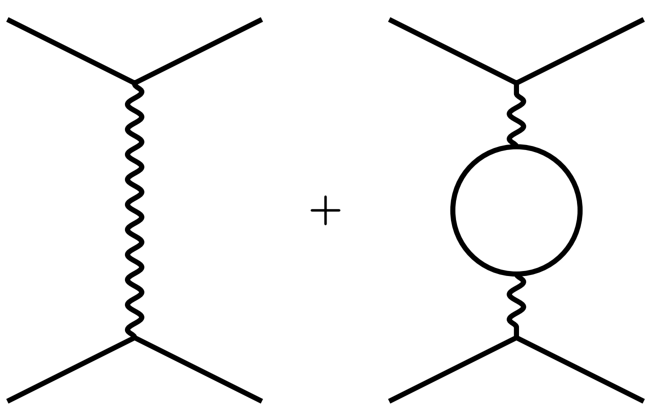
$$I_{\mu\nu}(q, m_B) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) I(q^2, m_B)$$

$$I(q^2, m_B) = \frac{ie_B^2}{2\pi^2} \int_0^1 \mathrm{d}z z(1-z) \int_0^\infty \frac{\mathrm{d}t}{t} \exp\left[it\left(q^2 z(1-z) - m_B^2 + i\varepsilon\right)\right]$$

- Pauli-Villars regularization: $I(q^2, m_B) \Rightarrow \bar{I}(q^2, m_B, M) \equiv I(q^2, m_B) - I(q^2, M)$, with very large M

$$\bar{I}(q^2, m_B, M) = \frac{ie_B^2}{2\pi^2} \int_0^1 \mathrm{d}z\, z(1-z) \ln\left[\frac{M^2}{m_B^2 - q^2 z(1-z)}\right]$$

- isolate divergent part (introducing renormalization scale μ)



$$-i\bar{I}(q^2, m_B, M) = \frac{e_B^2}{12\pi^2} \ln\frac{M^2}{\mu^2} - \frac{e_B^2}{2\pi^2} \int_0^1 \mathrm{d}z z(1-z) \ln\frac{m_B^2 - q^2 z(1-z)}{\mu^2}$$

$$-i\frac{g_{\mu\nu}}{q^2} \underbrace{e_B^2 \overbrace{\left(1 - \frac{e_B^2}{12\pi^2} \ln\frac{M^2}{\mu^2}\right)}^{Z_3}}_{e_R^2} \left[1 + \frac{e_B^2}{2\pi^2} \int_0^1 \mathrm{d}z z(1-z) \ln\frac{m_B^2 - q^2 z(1-z)}{\mu^2}\right]$$

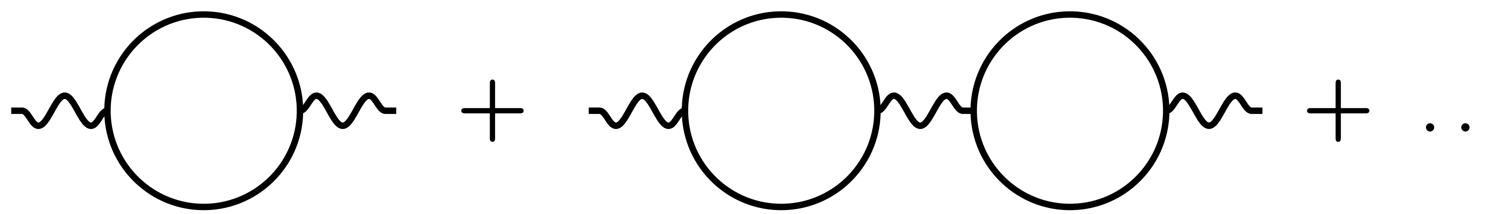
Running of α in QED

$$e_R^2 = e_B^2 \left(1 - \frac{e_B^2}{12\pi^2} \ln \frac{M^2}{\mu^2}\right) \qquad \Rightarrow \qquad \alpha_R(\mu) = \alpha_B \left(1 - \underbrace{\frac{2}{3\pi}}_{=\, b} \alpha_B \ln \frac{M}{\mu}\right)$$

- now, we would like to send $M \to \infty$
- however, this is impossible for finite $\alpha_R(\mu)$ at any fixed scale μ

quadratic equation: $D = 1 - 4b \ln(M/\mu)\, \alpha_R(\mu) < 0, \quad$ for $M \to \infty$

Landau's trick how to avoid quadratic eq. and irrational α and how to obtain more precise result



geometric series $\alpha_R(\mu) = \alpha_B \left(1 - b\alpha_B \ln \frac{M}{\mu} + b^2 \alpha_B^2 \ln^2 \frac{M}{\mu} + \ldots\right) = \frac{\alpha_B}{1 + b\alpha_B \ln \frac{M}{\mu}} \;\Rightarrow\; \alpha_B = \frac{\alpha_R(\mu)}{1 - b\alpha_R(\mu) \ln \frac{M}{\mu}}$

- still, we can't send $M \to \infty$
  - for any finite μ and non-zero, α_B we have $\alpha_R(\mu) \to 0$ **no electric charge (!)**
  - these were the reasons why Dirac and Landau did not trust QED completely

Running of α in QED

$$\boxed{\alpha_R(\mu) = \alpha_B \left(1 - b\,\alpha_B \ln\frac{M}{\mu}\right), \qquad b = \frac{2}{3\pi}}$$

or $\quad \alpha_R(\mu) = \frac{\alpha_B}{1+b\alpha_B \ln\frac{M}{\mu}} + \mathcal{O}(\alpha_B^2)$

- let's still try to eliminate M

$$\alpha_R(\mu_2) = \frac{\alpha_B}{1 + b\alpha_B \ln\frac{M}{\mu_2}} = \frac{\alpha_B}{1 + b\alpha_B \ln\frac{M}{\mu_1} + b\alpha_B \ln\frac{\mu_2}{\mu_1}} = \frac{\alpha_B}{\frac{\alpha_B}{\alpha_R(\mu_1)} + b\alpha_B \ln\frac{\mu_1}{\mu_2}} = \frac{\alpha_R(\mu_1)}{1 - b\alpha_R(\mu_1)\ln\frac{\mu_2}{\mu_1}}$$

- another trick how to get rid of M is to differenciate $\alpha_R(\mu)$

$$\frac{\partial\alpha_R(\mu)}{\partial\ln\mu} = b\alpha_B^2 + b_1\alpha_B^3 + \ldots = b\alpha_R(\mu)^2 + \bar{b}_1\alpha_R(\mu)^3 + \ldots$$

  - **beta function** (function of type $\partial g/\partial\ln\mu = \beta(g)$)
  - no explicit dependence on μ (**renormalization group equation**)

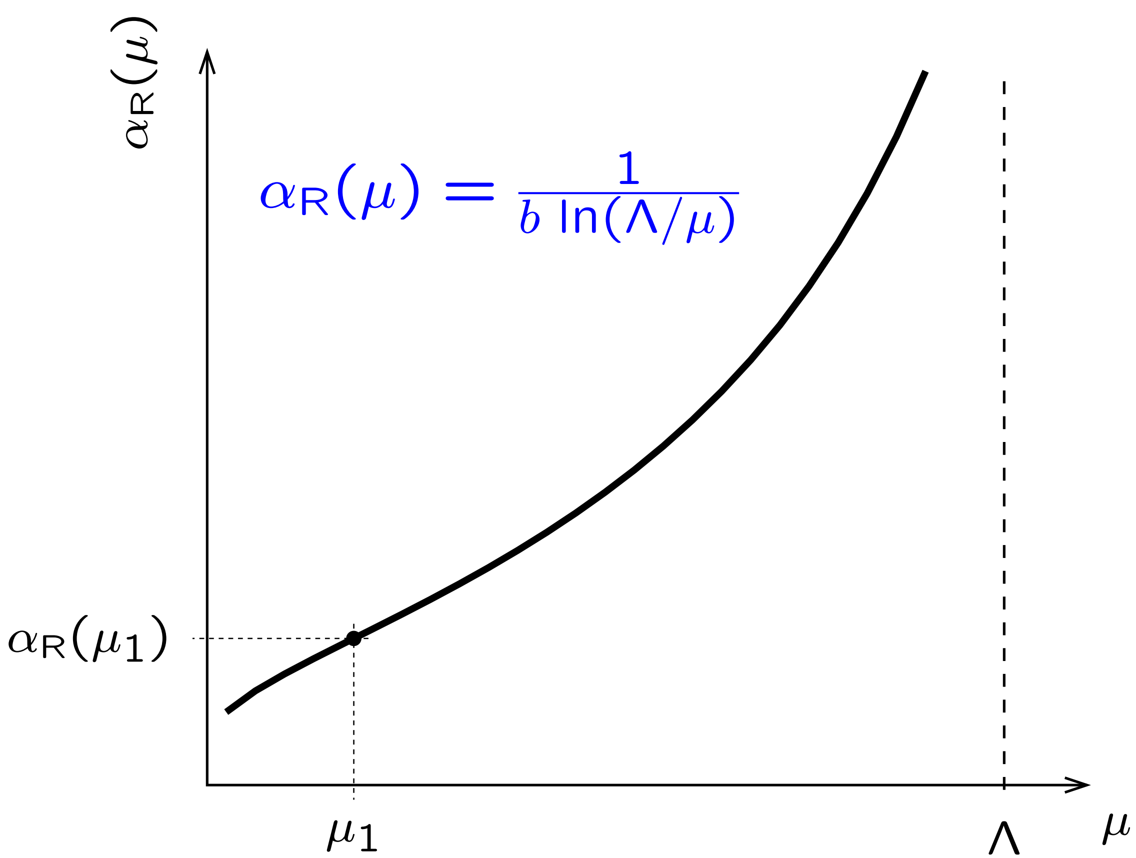
Running of α - 1-loop β function

$$\boxed{\frac{\partial \alpha_R(\mu)}{\partial \ln \mu} = b\alpha_R^2(\mu)}$$

- Non-linear differencial equation of the 1st order
- no explicit dependence of β function on μ

$$\frac{1}{\alpha_R^2}\,\mathrm{d}\alpha_R = b\,\mathrm{d}\ln\mu \quad \Rightarrow \quad -\frac{1}{\alpha_R(\mu)} = b\,\ln\mu + C$$

getting rid of boundary condition C : $\frac{1}{\alpha_R(\mu_2)} - \frac{1}{\alpha_R(\mu_1)} = b\ln\frac{\mu_1}{\mu_2} \quad \Rightarrow \quad \alpha_R(\mu_2) = \frac{\alpha_R(\mu_1)}{1 - b\alpha_R(\mu_1)\ln(\mu_2/\mu_1)}$



- for $b > 0$ and finite $\alpha_R(\mu_1) > 0$

  $\exists$ scale Λ at which $\alpha_R(\Lambda) \to \infty$ $\quad b\alpha_R(\mu_1)\ln(\Lambda/\mu_1) = 1$
- we can rewrite $C = -b\ln\Lambda$

$$\boxed{\alpha_R(\mu) = \frac{1}{b\ln\frac{\Lambda}{\mu}}}$$

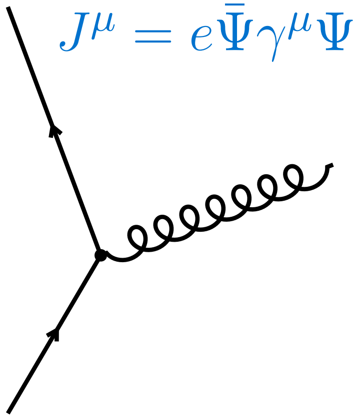
- electron - the lightest charged particle

$$\alpha(m_e) = \frac{1}{137} \quad \Rightarrow \quad \Lambda_{\text{Landau}} = m_e \exp[1/(b\alpha)] \approx 10^{280} m_e$$

- QED - good theory for all practical purposes

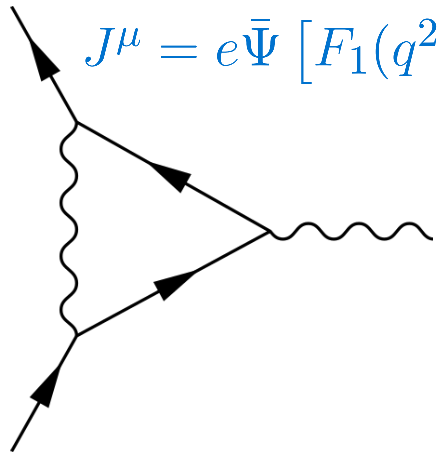
Test of pQED - electron magnetic moment

$J^\mu = e\bar{\Psi}\gamma^\mu\Psi$



$g = 2$

$J^\mu = e\bar{\Psi}\left[F_1(q^2)\gamma^\mu + \frac{i\kappa}{2m}F_2(q^2)\sigma^{\mu\nu}q_\nu\right]\Psi$



$a = \frac{\alpha}{2\pi}$ (App. B)

$$a = \frac{g-2}{g} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) + \ldots$$

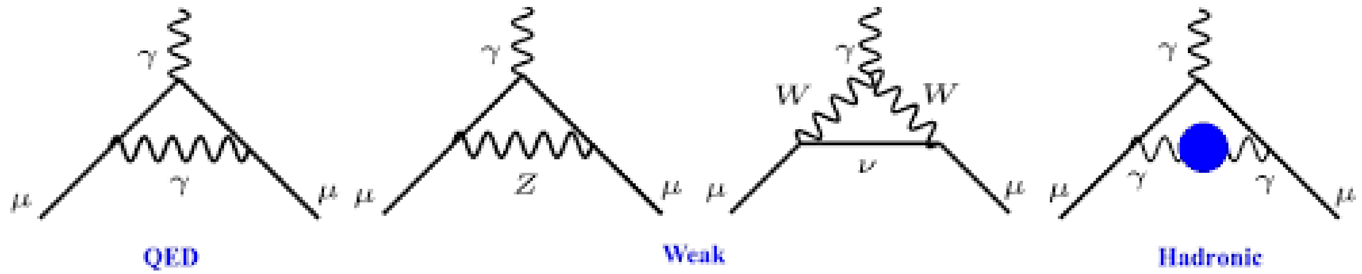
en.wikipedia.org, J. Bourjaily

- stringent test of perturbative QED (Nobel Prize 1965: S. Tomonaga, J. Schwinger, R. Feynman)
- today, we can calculate contributions to a up to α^5

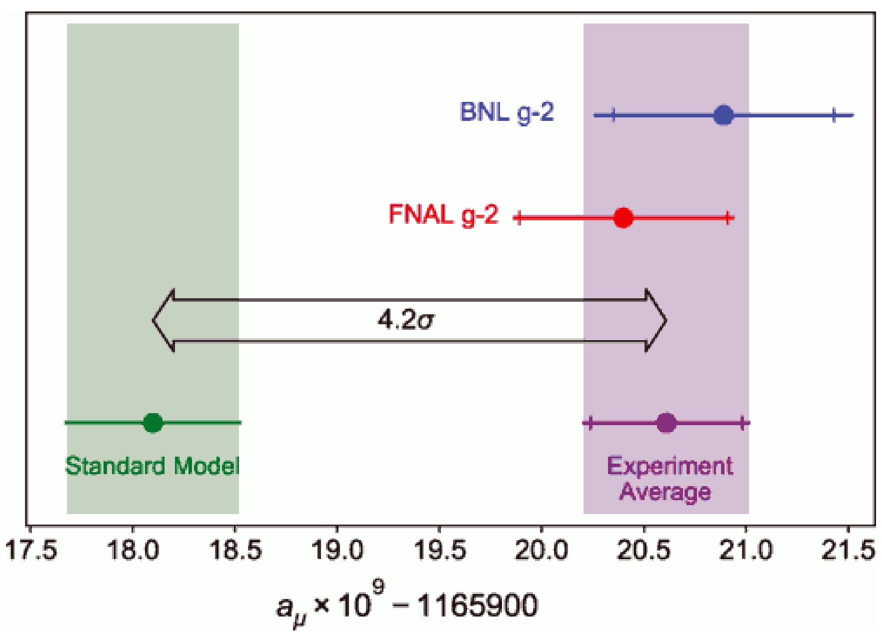
  $a_{exp} = 0.001\,159\,652\,180\,73\,(28) \quad \Rightarrow \quad \alpha^{-1} = 137.035\,999\,070\;(98)$
- independent measurement of α (atomic spectra): $\alpha^{-1} = 137.035\,998\,78\;(91)$
- very precise test of QED at level of **1:100 000 000 (!)**

Muon magnetic moment anomaly

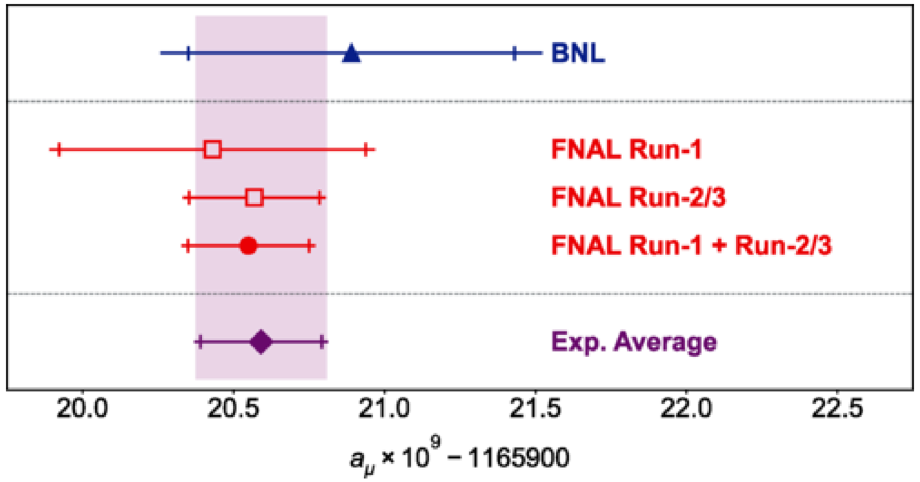
- interesting 3.7σ discrepancy between theory (KNT18) and experiment (E821 at BNL)
  - newly confirmed by experiment at Fermilab (2021 and 2023)
- potential hint of new physics (there could be new particles running in the loops)



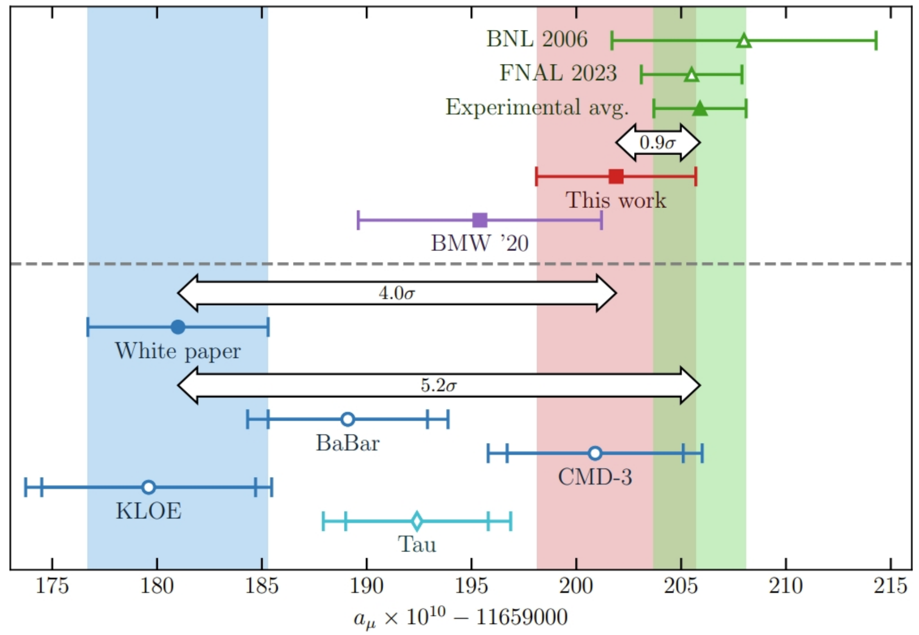
arXiv:1610.06587 [hep-ph]



Muon g-2 Coll., PRL 126 (2021) 141801



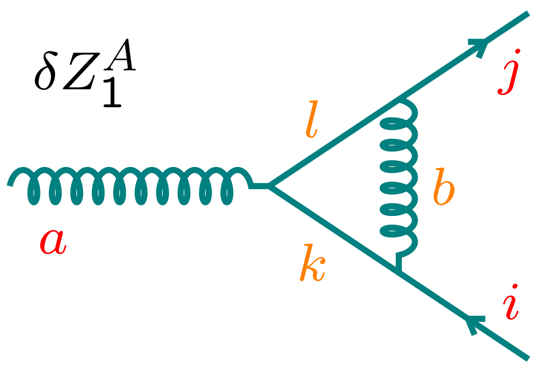
Muon g-2 Coll., PRL 131 (2023) 161802

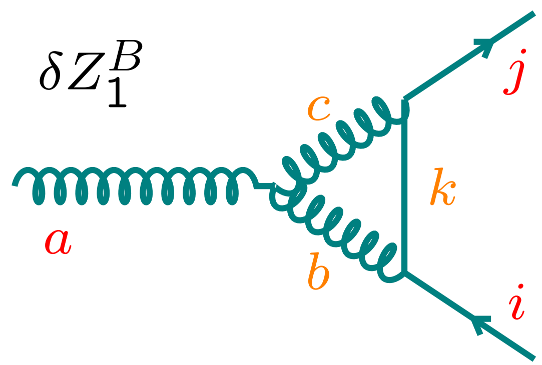


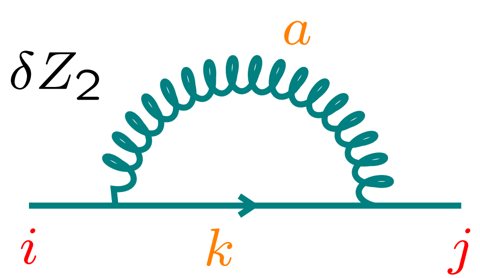
Z. Fodor for BMW+DMZ Coll., ICHEP 2024

- hadronic corrections - some contradictions between R-method (e^+e^- data) and lattice QCD
- no final conclusion yet but it looks like the discrepancy goes away with new lattice calculations

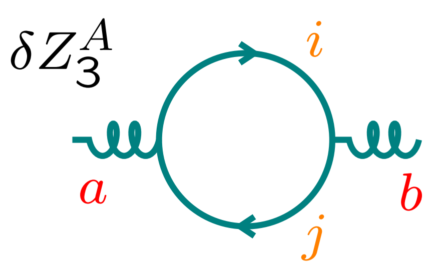
Running of α_S in SU(N) YM QFT

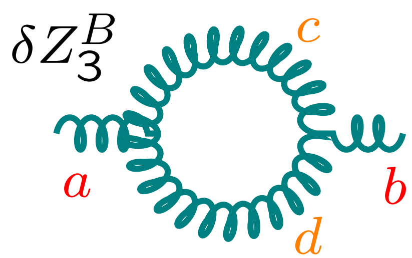


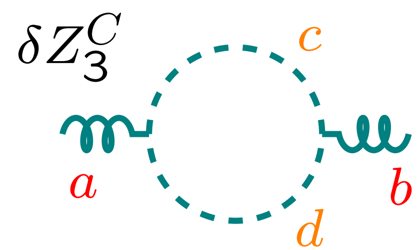




$$e_B = e_R \frac{Z_1}{Z_2 Z_3^{1/2}}$$







$$\alpha_S(\mu) = \alpha_S^{\text{bare}} \left(1 - \frac{n_f}{3\pi}\alpha_S \ln\frac{M}{\mu} + \frac{11 C_A}{6\pi}\alpha_S \ln\frac{M}{\mu} + \ldots\right)$$

$$T^a T^b = \frac{1}{6}\delta_{ab}\mathbf{1} + \frac{1}{2}(d_{abc} + i f_{abc})T^c$$

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}, \quad f_{abc} f_{abd} = C_A \delta_{cd}$$

- color factors for δZ terms

$$C_3^A = n_f T^b_{ji} T^a_{ij} = n_f \mathrm{Tr}(T^b T^a) = \tfrac{1}{2}\delta_{ab} n_f$$

$$C_3^B = C_3^C = f_{cad} f_{cdb} = -C_A \delta_{ab}$$

$$C_2 = (T^a T^a)_{ji} = C_F \delta_{ij}$$

$$C_1^B = f_{abc}(T^c T^b)_{ji} = \tfrac{1}{2} i f_{abc} f_{cbd} T^d_{ji} = -\tfrac{1}{2} i C_A T^a_{ji}$$

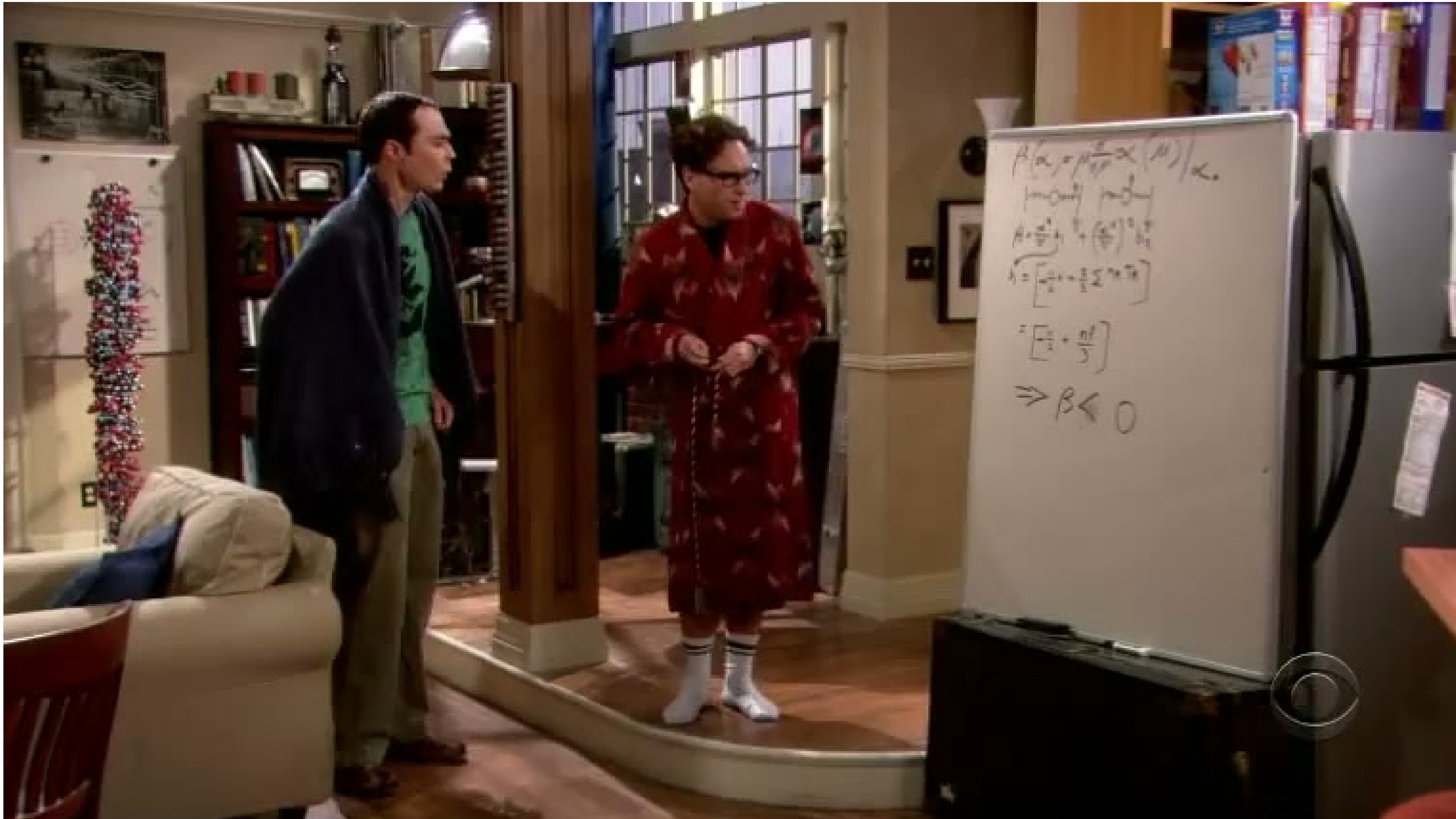
$$\begin{aligned} C_1^A &= (T^b T^a T^b)_{ji} = (T^b T^b T^a)_{ji} + (T^b [T^a, T^b])_{ji} \\ &= C_F T^a_{ji} + i f_{abc}(T^b T^c)_{ji} = (C_F - C_A/2) T^a_{ji} \end{aligned}$$

$$\boxed{b = \frac{2 n_f - 11 C_A}{6\pi}}$$

Gross, Wilczek, Politzer (1973) - the first to understand the significance of the result for the theory of strong interactions

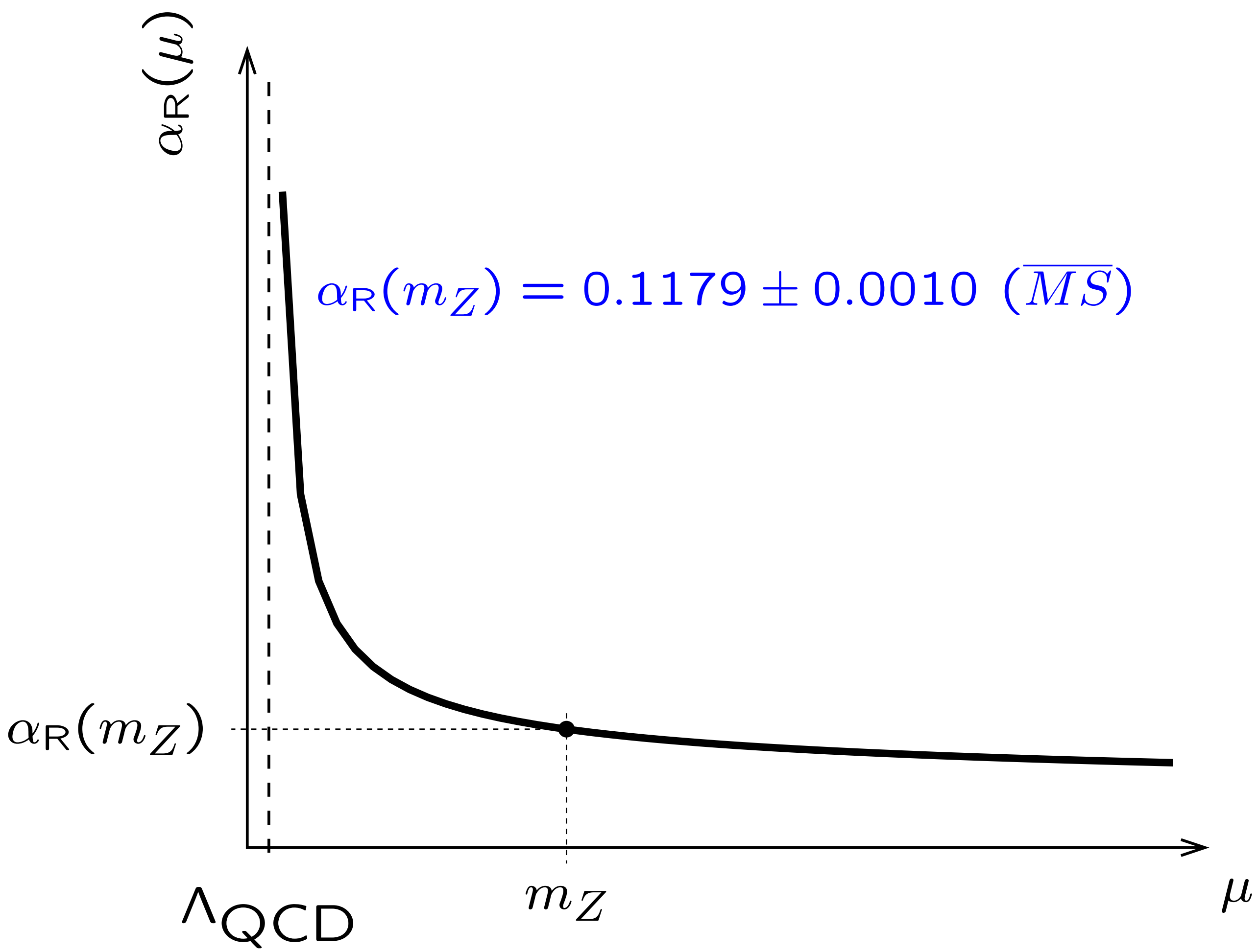
- not the first to discover Yang-Mills QFTs with asymptotic freedom - t'Hooft (1971), Khriplovich (1969)

Running of α_S in QCD



Big Bang Theory

Running of α_S in QCD



$$\alpha_S(\mu) = \frac{1}{-b \ln(\mu/\Lambda_{\mathrm{QCD}})}$$

- perfectly valid theory at large scales
  - limit $M \to \infty$ can be done
  - Dirac and Landau would be happy with such QFT

- the pole is still there, but this time at low scales - $\Lambda_{\mathrm{QCD}} \approx 200$ MeV
  - at right scale corresponding to about 1 fm - possible explanation for **confinment**

History of Asymptotic Freedom

1954 Yang & Mills study vector field theory with non-Abelian gauge invariance.

1965 Vanyashin & Terentyev compute vacuum polarization due to a massive charged vector field. In our notation, they found

$$\beta_0 = \frac{1}{12\pi}\left(-\frac{21}{2} = -11 + \frac{1}{2}\right)$$

- The $\frac{1}{2}$ comes from longitudinal polarization states (absent for massless gluons)
- They concluded that this result "... seems extremely undesirable".

1969 Khriplovich correctly computes the one-loop β-function in SU(2) Yang-Mills theory using the Coulomb gauge

$$\beta_0 = \frac{C_A}{12\pi}(-12 + 1 = -11)$$

- In Coulomb gauge the anti-screening (–12) is due to an instantaneous Coulomb interaction
- He did not make a connection with strong interactions

1971 't Hooft correctly computes the one-loop β-function for SU(3) gauge theory but does not publish it.

- He wrote it on the blackboard at a conference
- His supervisor (Veltman) told him it wasn't interesting
- 't Hooft & Veltman received the 1999 Nobel Prize for proving the *renormalizability* of QCD (and the whole Standard Model).

1972 Fritzsch & Gell-Mann propose that the strong interaction is an SU(3) gauge theory, later named QCD by Gell-Mann

1973 Gross & Wilczek, and independently Politzer, compute and publish the 1-loop β-function for QCD

$$\beta_0 = \frac{1}{12\pi}(2n_f - 11C_A)$$

$\Rightarrow$ 2004 Nobel Prize (now that 't Hooft has one anyway . . .)

1974 Caswell & Jones compute the 2-loop β-function for QCD

1980 Tarasov, Vladimirov & Zharkov compute the 3-loop β-function for QCD

1997 van Ritbergen, Vermaseren & Larin compute the 4-loop β-function for QCD ($\sim 50,000$ Feynman diagrams)

". . . We obtained in this way the following result for the 4-loop beta function in the $\overline{\mathrm{MS}}$-scheme:

$$q^2 \frac{\partial a_s}{\partial q^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

where $a_s = \alpha_S / 4\pi$ and . . .

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f$$

$$\beta_2 = \frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f - \frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2$$

$$\begin{aligned}
\beta_3 = {} & C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + C_A^3 T_F n_f\left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\
& + C_A^2 C_F T_F n_f\left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f\left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) \\
& + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2\left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) \\
& + C_A C_F T_F^2 n_f^2\left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) + \frac{424}{243}C_A T_F^3 n_f^3 + \frac{1232}{243}C_F T_F^3 n_f^3 \\
& + \frac{d_A^{abcd}d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + n_f\frac{d_F^{abcd}d_A^{abcd}}{N_A}\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\
& + n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right)
\end{aligned}$$

- β_0 and β_1 universal
- β_2 and higher depend on choice of renormalization scheme
- coupling is parameter not an observable (!)

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202\cdots$) and the colour factors for SU(N) are

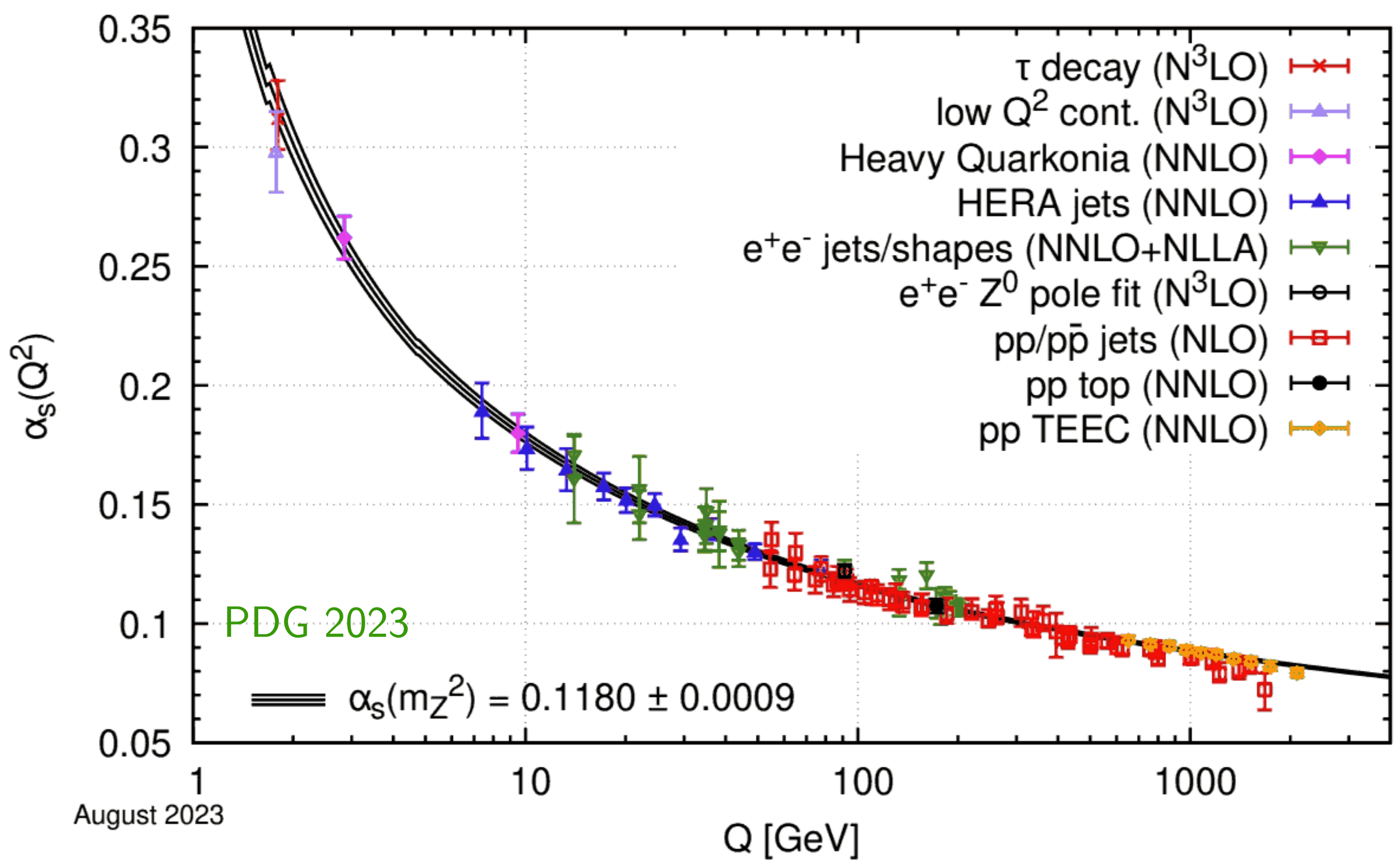
$$T_F = \frac{1}{2}, \quad C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad \frac{d_A^{abcd}d_A^{abcd}}{N_A} = \frac{N^2(N^2+36)}{24},$$

$$\frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N(N^2+6)}{48}, \quad \frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N^4-6N^2+18}{96N^2}$$

Substitution of these colour factors for $N = 3$ yields the following numerical results for QCD:

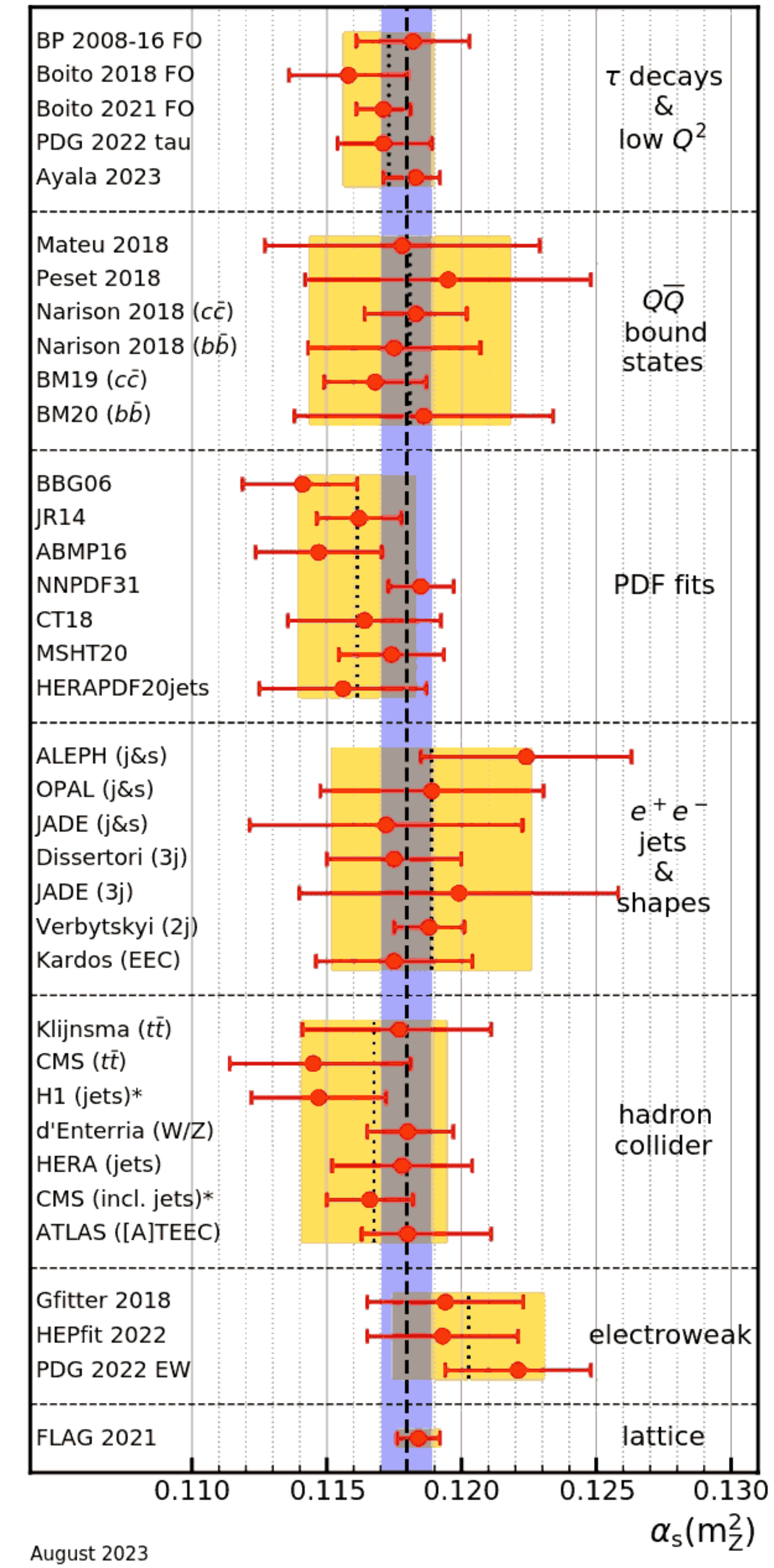
$$\begin{aligned}
\beta_0 &\approx 11 - 0.66667 n_f \\
\beta_1 &\approx 102 - 12.6667 n_f \\
\beta_2 &\approx 1428.50 - 279.611 n_f + 6.01852 n_f^2 \\
\beta_3 &\approx 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3
\end{aligned}$$

Running of α_S - experiment



PDG'13: $\alpha_S(M_Z) = 0.1184 \pm 0.0007$ $(\overline{MS})$

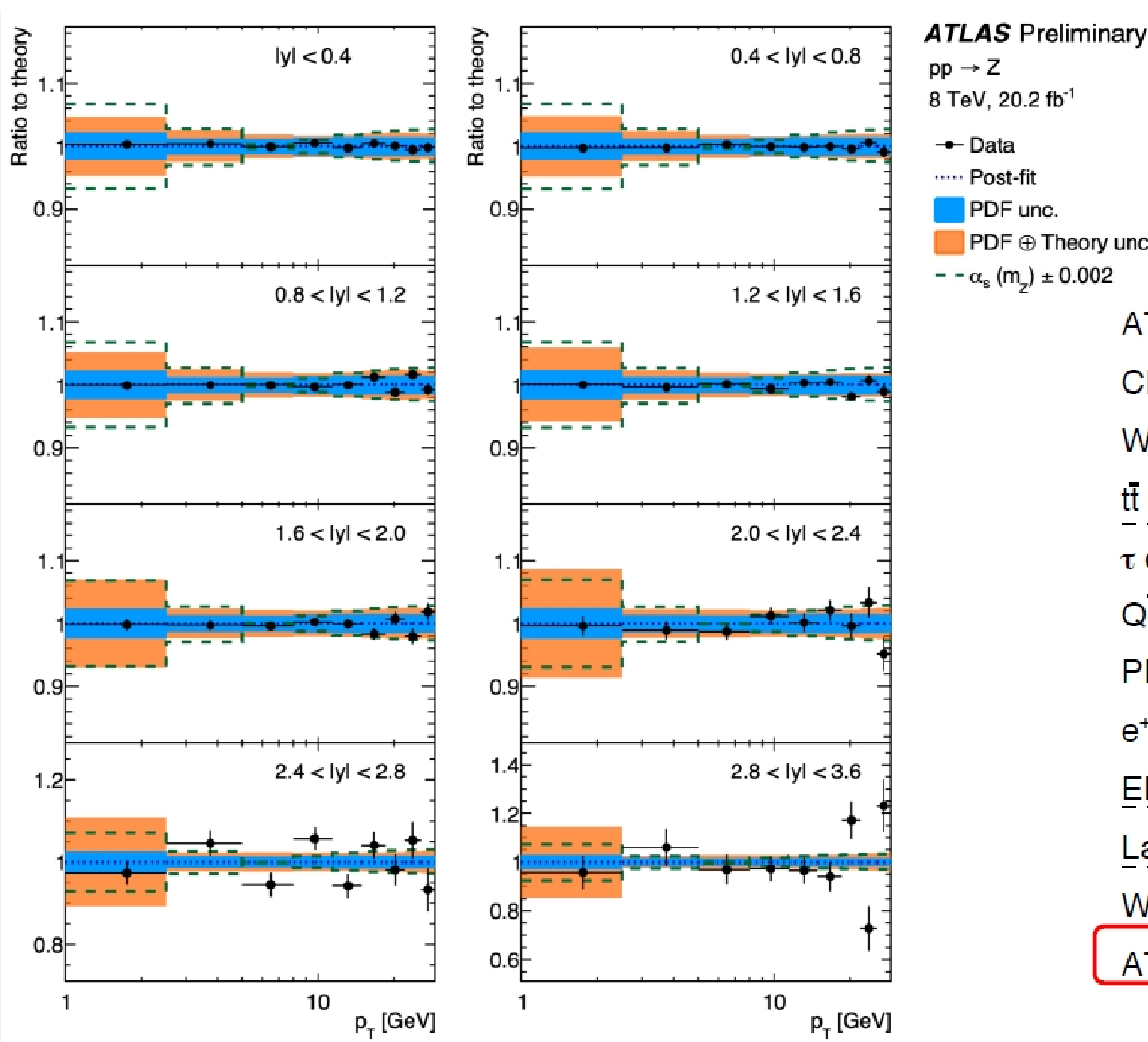
- at HL-LHC and FCC, α_S together with PDF become dominant theoretical uncertainties for many interesting processes (including H production)



Strong coupling strength measurement [physics briefing]

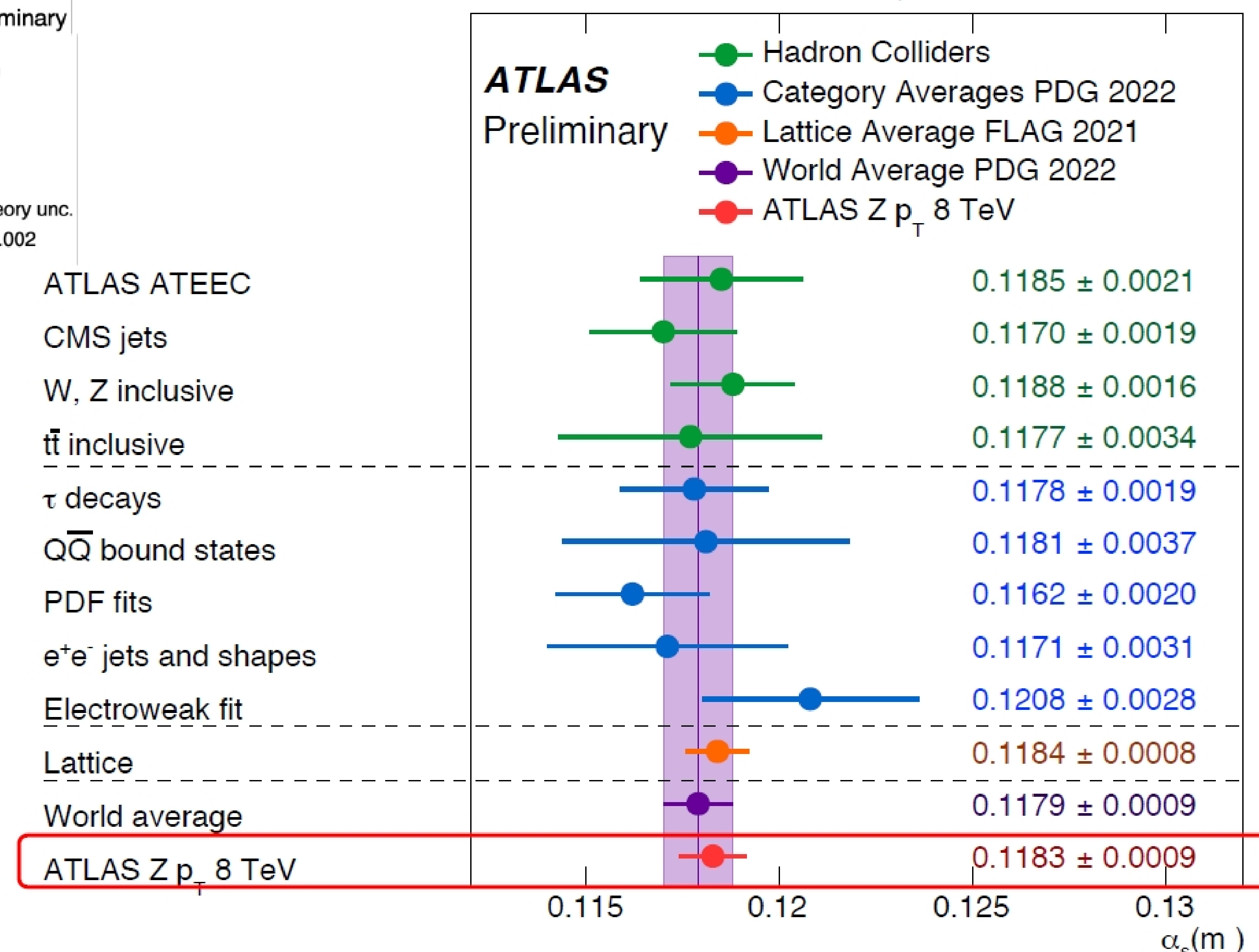
Most precise determination of $\alpha_S(m_Z)$ from precise measurement & prediction of full-PS Z p_T spectrum at 8 TeV

- Virtual, Sudakov and real gluon emission known to N^{3}LL+N^{3}LO and approximate N^{4}LL+N^{3}LO



Ratios of double-differential Z p_T and rapidity measured cross sections to fit predictions

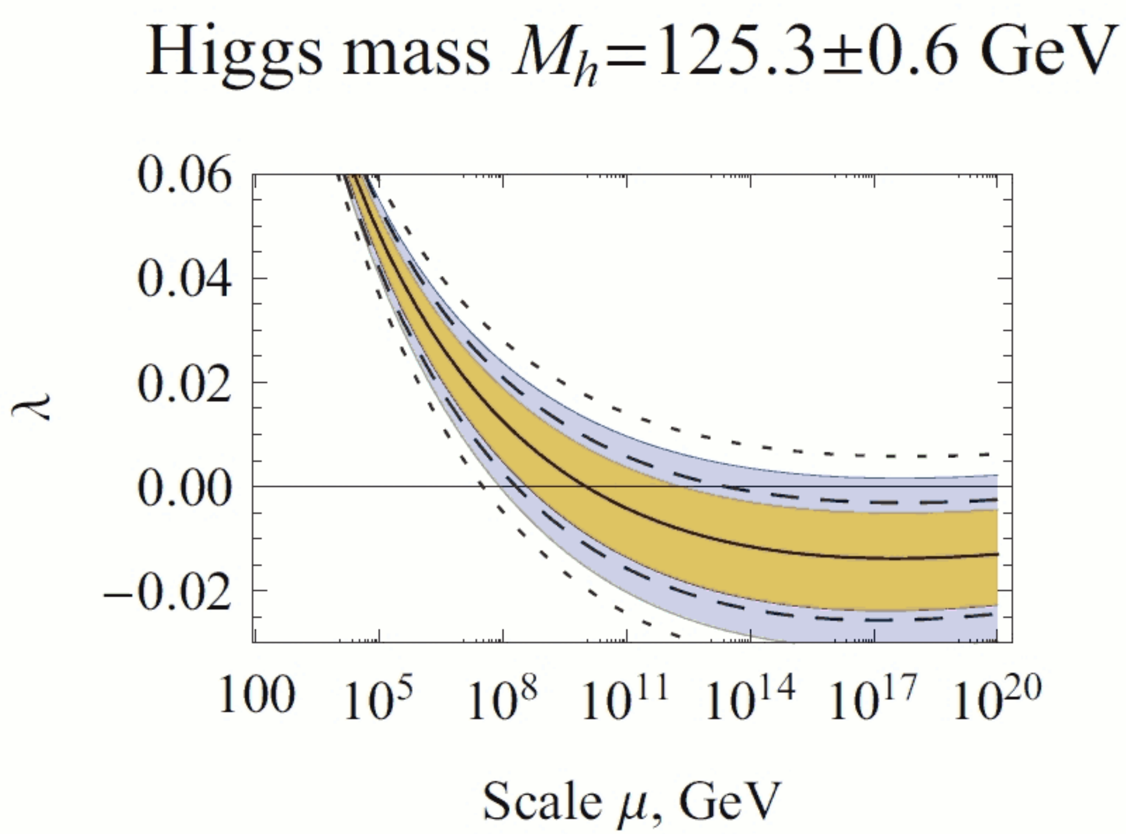
ATLAS-CONF-2023-015, ATLAS-CONF-2023-013

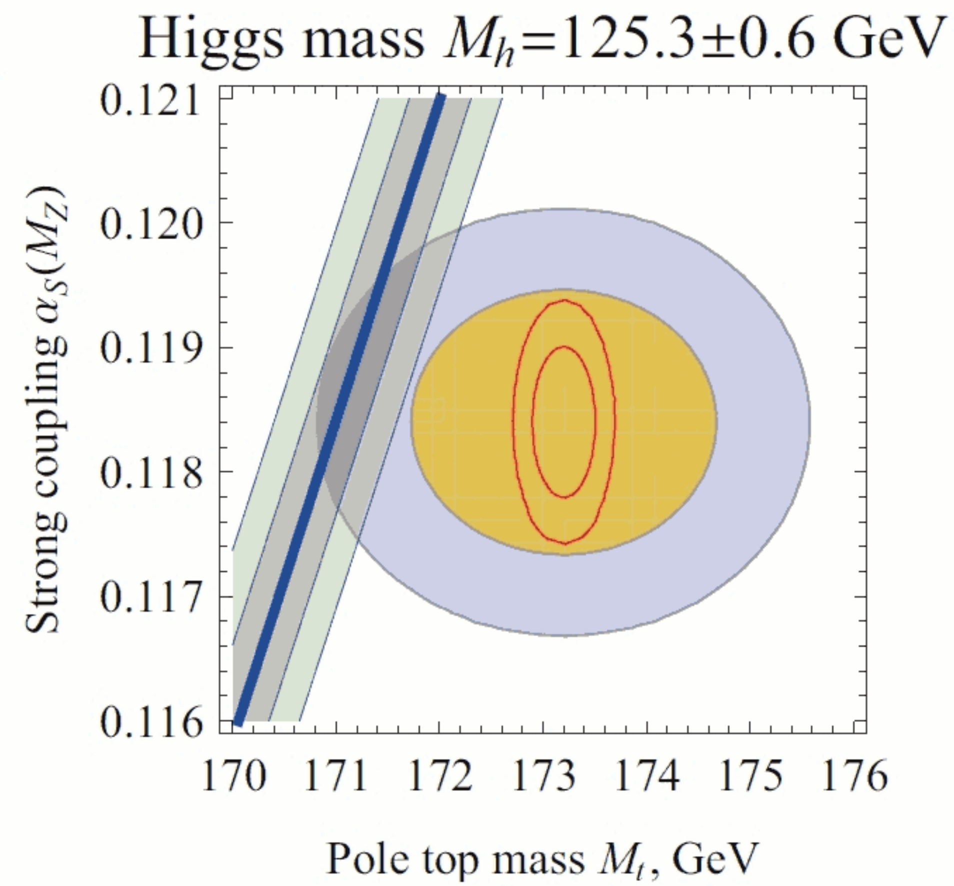


43

from A. Hoecker, ATLAS Week 2023

We do not know which possibility is realised!





errors in y_t: theory + experiment

Tevatron: $M_t = 173.2 \pm 0.51 \pm 0.71$ GeV

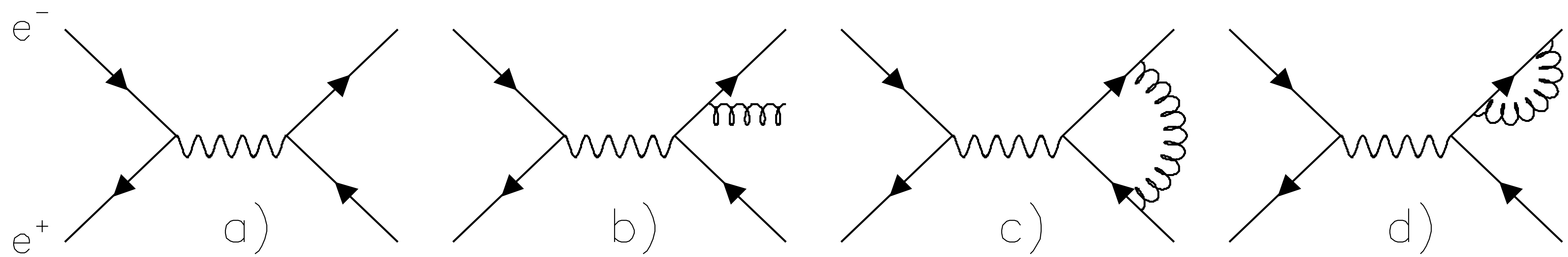
ATLAS and CMS: $M_t = 173.4 \pm 0.4 \pm 0.9$ GeV

$\alpha_s = 0.1184 \pm 0.0007$

from EPS'13 Bezrukov slides

Stockholm, July 23, 2013 – p. 9

Renormalization in QCD



$$R(Q) \equiv \frac{\sigma_{e^+e^-\to\text{hadrons}}(Q)}{\sigma_{e^+e^-\to\mu^+\mu^-}(Q)} = \overbrace{\left(3\sum_i e_i^2\right)}^{QPM}[1+r(Q)]$$

- QCD provides higher order corrections in power series of $a \equiv \alpha_S/\pi$

$$r(Q) = a(\mu/\Lambda)\left[r_0 + r_1(\mu/Q)a(\mu/\Lambda) + r_2(\mu/Q)a^2(\mu/\Lambda) + \ldots\right], \quad \text{with } r_0 = 1$$

- a dependence on μ is given by beta function

$$\frac{\mathrm{d}a(\mu, c_i)}{\mathrm{d}\ln\mu} = ba^2(\mu, c_i)\left[1 + ca(\mu/\Lambda) + c_2a^2(\mu/\Lambda)\right]$$

  b and c unique, c_2 arbitrary (!) - their choice defines renormalizaion scheme (or RS defines thier values)

- r_i depend on RS, $r_i(\mu/Q, c_i, RS)$, in such a way that $r(Q)$ does not depend on μ and RS
- at fixed order, dependence on μ remains

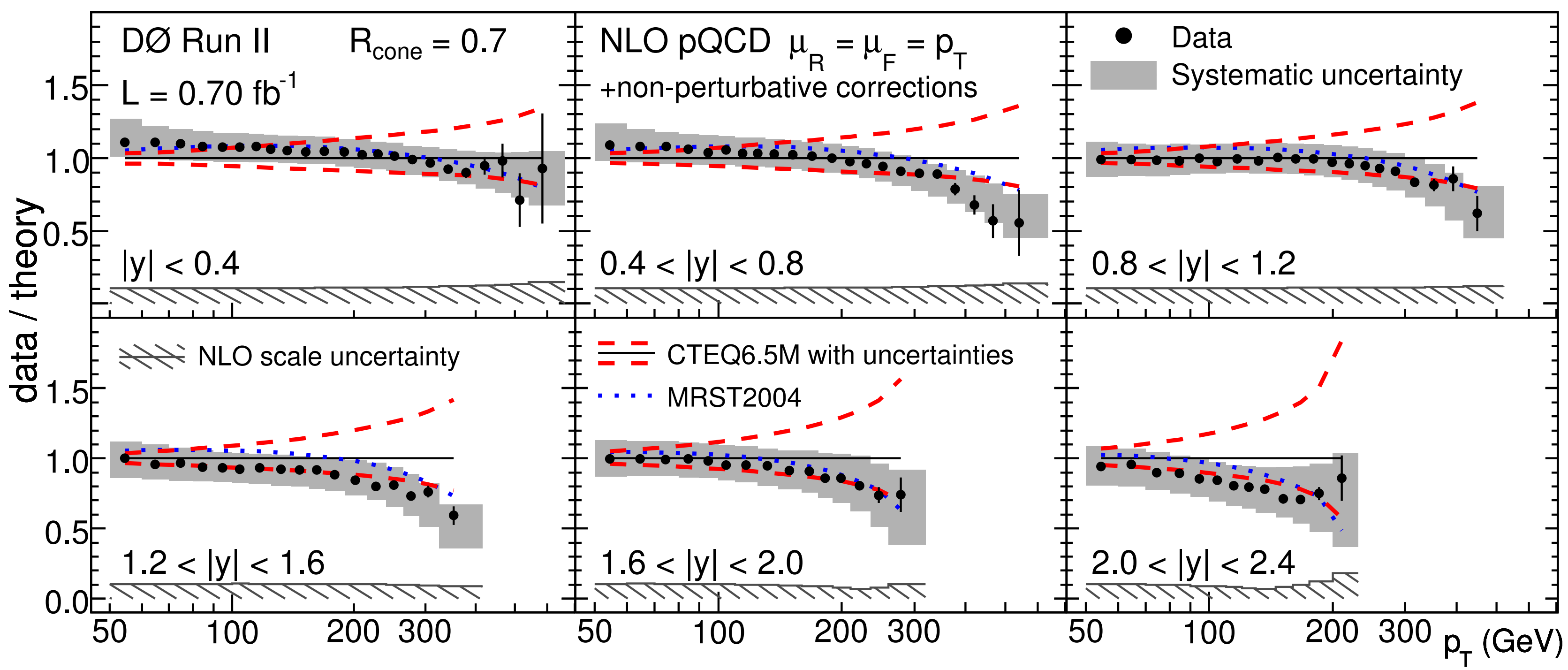
- typical choice of μ: $\mu \approx Q$ (scale of the process), higher order correction have typically $\ln(\mu/Q)$ behaviour

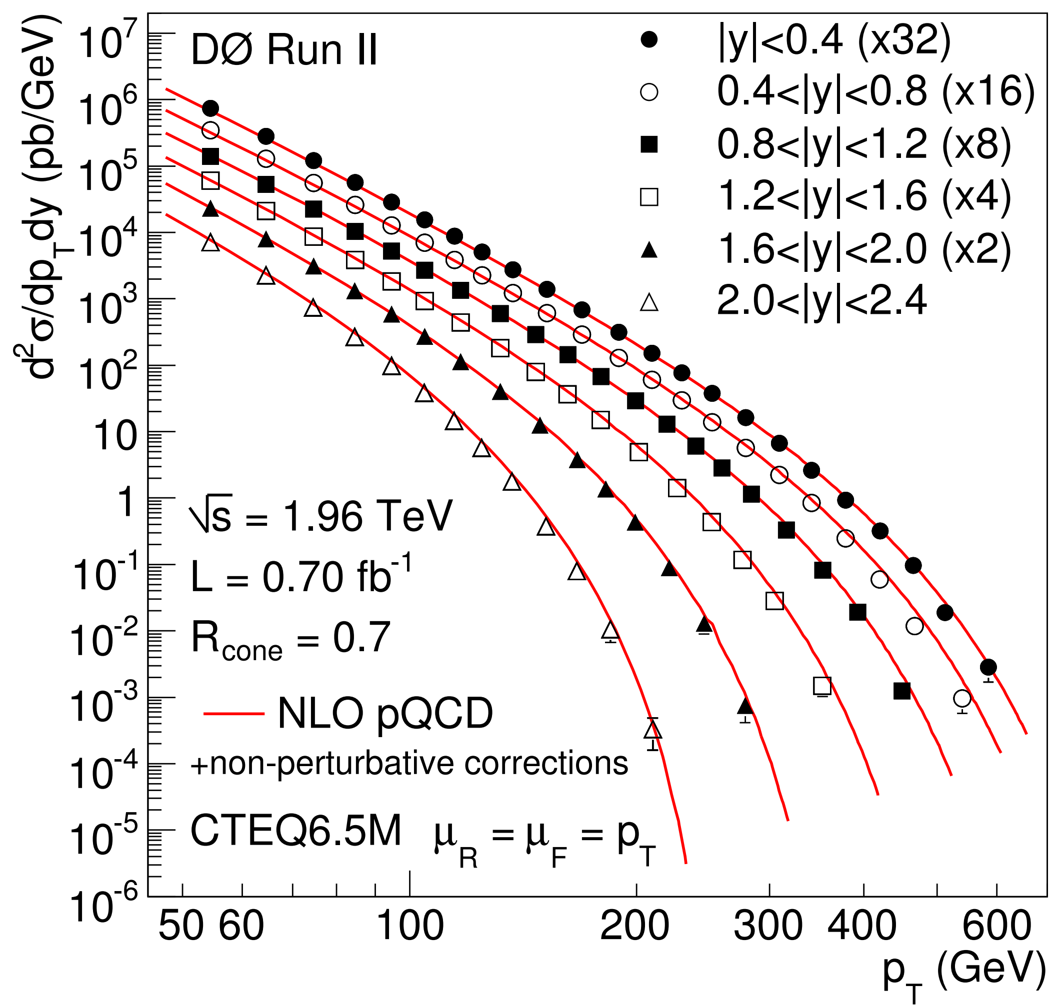
Test QCD - jet production - 2008 (DØ)

- "Rutherford experiment" in $p\bar{p}$ at $\sqrt{s} = 1.96$ TeV

$$\frac{d\sigma_{\mathrm{QPM}}}{dp_T} = \sum_{a,b,c,d} \int dx_1 dx_2\, d_a(x_1) d_b(x_2) \sigma_{ab\to cd} \left[\delta(p_\mathrm{T} - p_\mathrm{T}^c) + \delta(p_\mathrm{T} - p_\mathrm{T}^d)\right]$$

- highest momentum transfer with $p_\mathrm{T} \approx 600$ GeV probing QCD at disctance scales of $\Delta x \sim 10^{-3}$ fm





PRL 101 (2008) 062001

- **scale uncertainty**

  μ_R and μ_F varied up and down by factor of 2
- measure of ignorance on higher order terms

For reading and for exercises

- Jiri Chyla's textbook chapters
  - 8.1 (Infinities in perturbative calculations)
  - 8.3.5 (Renormalization at one loop level and the technique of counterterms)
  - 8.4 (Renormalization in QCD)
- Exercise
  - textbook chapter 8.2.1 (Electric charge renormalization)
- Reading for further study
  - 8.3 - Elements of dimensional regularization