Quarks, Partons, and Quantum Chromodynamics

Lecture 3

Groups and symmetries - group SU(3)

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SU(3) generators - color matrices T_a

$SU(3)$ group – a set of 3×3 unitary (U) complex matrices with unit determinant (S - special)

- 3×3 matrices form also representation on 3-dim $\mathcal{H}$
- generators are 3×3 Hermitian traceless matrices

$$\begin{pmatrix} a & c-id & e-if \\ c+id & b & g-ih \\ e+if & g+ih & -a-b \end{pmatrix} \qquad \det e^A = e^{Tr(A)}$$

- 8 generators - Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- **color matrices** $T_a = \frac{1}{2}\lambda_a$ satisfy

$$\mathrm{Tr}(T_a T_b^+) = \frac{1}{2}\delta_{ab}$$

- in analogy with Pauli matrices

$$\boxed{T_a \cdot T_b = \frac{1}{6}\delta_{ab}\mathbb{1} + \frac{1}{2}(d_{abc} + i f_{abc}) T_c}$$

Exercise 3.3

- $d_{abc} \in \mathbb{R}$ fully symmetric
- defined via anti-commutator

$$\{T_a, T_b\} = \frac{1}{3}\delta_{ab}\mathbb{1} + d_{abc} T_c$$

Group SU(N)

- $N - 1$ real numbers on diagonal
- $N^2 - N$ real numbers for off-diagonal elements

$\Rightarrow$ $N^2 - 1$ generators

SU(3) fundamental representation and roots

- SU(3) rank is $r = 2$:

$$H_1 \equiv T_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 \equiv T_8 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

- 3 eigen states forming basis of $\mathcal{H}$:

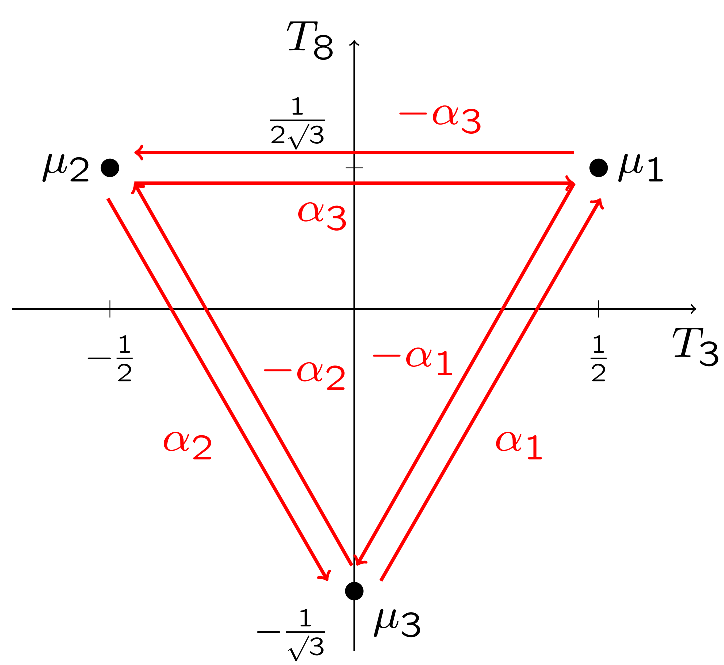
$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T_3\,|e_1\rangle = \frac{1}{2}\,|e_1\rangle, \quad T_8\,|e_1\rangle = \frac{1}{2\sqrt{3}}\,|e_1\rangle \quad \Rightarrow \quad \mu_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$$

- we find for $|e_2\rangle$ and $|e_3\rangle$

$$\mu_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$$

$$\mu_3 = \left(0, -\frac{1}{\sqrt{3}}\right)$$



- 8-2=6 roots: $\pm\boldsymbol{\alpha}_1$, $\pm\boldsymbol{\alpha}_2$, $\pm\boldsymbol{\alpha}_3$

$$\boldsymbol{\alpha}_1 = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\boldsymbol{\alpha}_2 = \boldsymbol{\mu}_3 - \boldsymbol{\mu}_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$\boldsymbol{\alpha}_3 = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = (1, 0)$$

- note that $|\boldsymbol{\alpha}_i| = 1$ ... consequence of norm. choice $\lambda = \frac{1}{2}$
- corresponding 6 shifting operators could be guessed

$$E_{\alpha_3} = \frac{T_1 + iT_2}{\sqrt{2}}, \qquad E_{-\alpha_3} = E^+_{\alpha_3} = \frac{T_1 - iT_2}{\sqrt{2}}$$

- in analogy

$$E_{\alpha_1} = \frac{T_4 + iT_5}{\sqrt{2}}, \qquad E_{-\alpha_1} = \frac{T_4 - iT_5}{\sqrt{2}}$$

$$E_{\alpha_2} = \frac{T_6 - iT_7}{\sqrt{2}}, \qquad E_{-\alpha_2} = \frac{T_6 + iT_7}{\sqrt{2}}$$

Exercise 3.4

Roots and Weights

$\times\ E_{\alpha}^{p+1} |\mu\rangle = 0$

$\vdots$

- $|\mu + p\alpha\rangle \propto E_{-\alpha}^{p} |\mu\rangle$

$\vdots$

- $|\mu + \alpha\rangle \propto E_{\alpha} |\mu\rangle$

$\uparrow\ E_{\alpha}$

(p, q) • $|\mu\rangle$

$\downarrow\ E_{-\alpha}$

- $|\mu - \alpha\rangle \propto E_{-\alpha} |\mu\rangle$

$\vdots$

- $|\mu - q\alpha\rangle \propto E_{-\alpha}^{q} |\mu\rangle$

$\downarrow\ E_{-\alpha}$

$\times\ E_{-\alpha}^{q+1} |\mu\rangle = 0$

- Prop. 2: irred. rep. of compact groups are of finite dim.

$\Rightarrow$ state $|\mu\rangle$ can be characterized by two integers

p: number of times, the state can be risen with E_{α}

q: number of times, the state can be lowered with $E_{-\alpha}$

- in analogy with SU(2)

$$E_{\alpha} |\mu\rangle = N_{\alpha,\mu} |\mu + \alpha\rangle \qquad E_{-\alpha} |\mu\rangle = N_{-\alpha,\mu} |\mu - \alpha\rangle$$

$$N_{-\alpha,\mu} N_{-\alpha,\mu}^{*} = \langle\mu| E_{\alpha} E_{-\alpha} |\mu\rangle = \langle\mu| \overbrace{[E_{\alpha}, E_{-\alpha}]}^{\alpha_i H_i} |\mu\rangle + \langle\mu| E_{-\alpha} E_{\alpha} |\mu\rangle$$
$$= \alpha \cdot \mu + N_{\alpha,\mu} N_{\alpha,\mu}^{*}$$

however

$$N_{-\alpha,\mu} = \langle\mu - \alpha| E_{-\alpha} |\mu\rangle \quad \Rightarrow \quad N_{-\alpha,\mu}^{*} = \langle\mu| \overbrace{E_{\alpha}}^{E_{\alpha}^{+} = E_{-\alpha}} |\mu - \alpha\rangle = N_{\alpha,\mu-\alpha}$$

$$\boxed{\alpha \cdot \mu = |N_{-\alpha,\mu}|^2 - |N_{\alpha,\mu}|^2 = |N_{-\alpha,\mu}|^2 - |N_{-\alpha,\mu+\alpha}|^2}$$

Master formula

$\times\ E_{\alpha}^{p+1}\,|\mu\rangle = 0$

$\vdots$

- $|\boldsymbol{\mu} + p\boldsymbol{\alpha}\rangle \propto E_{-\boldsymbol{\alpha}}^{p}\,|\boldsymbol{\mu}\rangle \qquad \boldsymbol{\alpha}\cdot(\boldsymbol{\mu}+p\boldsymbol{\alpha}) = |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}+p\alpha}|^2 - \overbrace{|N_{-\alpha,\boldsymbol{\mu}+(p+1)\alpha}|^2}^{=0}$

$\vdots \qquad \vdots$

- $|\boldsymbol{\mu}+\boldsymbol{\alpha}\rangle \propto E_{\boldsymbol{\alpha}}\,|\boldsymbol{\mu}\rangle \qquad \boldsymbol{\alpha}\cdot(\boldsymbol{\mu}+\boldsymbol{\alpha}) = |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}+\alpha}|^2 - |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}+2\boldsymbol{\alpha}}|^2$

$\uparrow\ E_{\boldsymbol{\alpha}}$

(p,q) • $|\boldsymbol{\mu}\rangle \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\mu} = |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}}|^2 - |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}+\boldsymbol{\alpha}}|^2$

$\downarrow\ E_{-\boldsymbol{\alpha}}$

- $|\boldsymbol{\mu}-\boldsymbol{\alpha}\rangle \propto E_{-\boldsymbol{\alpha}}\,|\boldsymbol{\mu}\rangle \qquad \boldsymbol{\alpha}\cdot(\boldsymbol{\mu}-\boldsymbol{\alpha}) = |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}-\alpha}|^2 - |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}}|^2$

$\vdots \qquad \vdots$

- $|\boldsymbol{\mu}-q\boldsymbol{\alpha}\rangle \propto E_{-\boldsymbol{\alpha}}^{q}\,|\boldsymbol{\mu}\rangle \qquad \boldsymbol{\alpha}\cdot(\boldsymbol{\mu}-q\boldsymbol{\alpha}) = \overbrace{|N_{-\alpha,\boldsymbol{\mu}-q\alpha}|^2}^{=0} - |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}-(q-1)\boldsymbol{\alpha}}|^2$

$\downarrow\ E_{-\boldsymbol{\alpha}}$

$\times\ E_{-\alpha}^{q+1}\,|\mu\rangle = 0$

$$\sum:\ \frac{p+q+1}{2}\left[2\boldsymbol{\alpha}\cdot\boldsymbol{\mu} + (p-q)\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\right] = 0$$

$$\boxed{\boldsymbol{\alpha}\cdot\boldsymbol{\mu} = |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}}|^2 - |N_{-\boldsymbol{\alpha},\boldsymbol{\mu}+\boldsymbol{\alpha}}|^2}$$

- summing over states from $|\boldsymbol{\mu}\rangle$ to $|\boldsymbol{\mu}+p\boldsymbol{\alpha}\rangle$

$$|N_{-\boldsymbol{\alpha},\boldsymbol{\mu}}|^2 = \frac{p+1}{2}\left(2\,\boldsymbol{\alpha}\cdot\boldsymbol{\mu} + p\,\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\right)$$

- summing over all $p+q+1$ states

$$\boxed{\frac{2\,\boldsymbol{\alpha}\cdot\boldsymbol{\mu}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = q-p}$$

Master formula

- nontrivial relation between weights and roots

Trick with adj. repr. - for any 2 roots $\boldsymbol{\alpha},\boldsymbol{\beta}$

$$\frac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = q-p = m \qquad \frac{2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} = q'-p' = m'$$

$$\Rightarrow \quad \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{|\boldsymbol{\alpha}|^2|\boldsymbol{\beta}|^2} = \cos^2\vartheta_{\alpha,\beta} = \frac{m\,m'}{4} = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1$$

- only certain angles allowed between roots

Root composition

- SU(3) roots: $\pm\boldsymbol{\alpha}_1$, $\pm\boldsymbol{\alpha}_2$, $\pm\boldsymbol{\alpha}_3$

$$\boldsymbol{\alpha}_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \boldsymbol{\alpha}_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad \boldsymbol{\alpha}_3 = (1, 0)$$

- note that $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1$
- order roots/weights

> Vector $\boldsymbol{\mu}$ is positive ($\boldsymbol{\mu} > 0$) if and only if its first non-zero element is positive

$$\boldsymbol{\mu}_1 > \boldsymbol{\mu}_2 \quad \Leftrightarrow \quad (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) > 0$$

- we can characterize irreducible representation $\mathcal{D}$ by the state with the highest weight
- concept of positive roots - SU(3) has 3 of them

> **Simple root** - positive root that cannot be written as sum of 2 positive roots

- SU(3) has 2 positive roots: $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$

- Jacobi identity

$$[H_i, [E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]] = -[E_{\boldsymbol{\beta}}, \underbrace{[H_i, E_{\boldsymbol{\alpha}}]}_{\alpha_i E_\alpha}] - [E_{\boldsymbol{\alpha}}, \underbrace{[E_{\boldsymbol{\beta}}, H_i]}_{-\beta_i E_\beta}] = (\alpha_i + \beta_i)[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]$$

- $[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]$ satisfy the same relation as shifting operator
- three cases
  1) $\boldsymbol{\alpha} + \boldsymbol{\beta} = 0$: $[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \alpha_i H_i$ did you prove it?
  2) $\boldsymbol{\alpha} + \boldsymbol{\beta} (\neq 0)$ is not root $\Rightarrow [E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] = 0$
  3) $\boldsymbol{\alpha} + \boldsymbol{\beta} (\neq 0)$ is root $\Rightarrow [E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] \propto E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$

> Prop 4: difference of two simple roots is not a root

> Prop 5: simple roots are lineary independent

> Prop 6: Number of simple roots of simple Lie algebra is equal to the rank of algebra

Simple alg. - no generator that would commute with all alg. members

- try to prove them
  - hint for Prop. 5, two simple roots $\boldsymbol{\alpha}, \boldsymbol{\beta}$ satistfy $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \leq 0$

in adj. rep. : $E_{-\boldsymbol{\alpha}} |\boldsymbol{\beta}\rangle = \big|[E_{-\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}]\big\rangle = 0 \Rightarrow q = 0 : \dfrac{2\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}} = -p \leq 0$

Fundamental weights and representations

- finite size $\Rightarrow$ irr. repr. $\mathcal{D}$ can be characterized by the weight of a state with maximal weight $\boldsymbol{\mu}_{\text{max}}$

$$\frac{2\,\boldsymbol{\alpha}_i\cdot\boldsymbol{\mu}_{\text{max}}}{|\boldsymbol{\alpha}_i|^2} = q_i - p_i = q_i \;(\geq 0)$$

$\Rightarrow$ instead of r (rank) real numbers ($\boldsymbol{\mu}_{\text{max}}$) we can characterize $\mathcal{D}$ by r non-negative whole numbers q_i, which specify how many times we can move from $|\boldsymbol{\mu}_{\text{max}}\rangle$ in direction $-\boldsymbol{\alpha}_i$ corresponding to simple root $\boldsymbol{\alpha}_i$

(Prop. 6 says that number of simple roots is equal to rank)

Fundamental representation $\mathcal{D}_F^{(i)}$ corresponding to the simple root $\boldsymbol{\alpha}_i$ is repr. $\mathcal{D}(q_1,\ldots,q_r)$ with $q_j=\delta_{ij}$.

Fundamental weight $\boldsymbol{\mu}_F^{(i)}$ is maximal weight of $\mathcal{D}_F^{(i)}$.

$$\frac{2\,\boldsymbol{\alpha}_j\cdot\boldsymbol{\mu}_F^{(i)}}{|\boldsymbol{\alpha}_j|^2} = q_j = \delta_{ij}$$

- maximal weight of $\mathcal{D}(q_1,\ldots,q_r)$ is from linearity

$$\boldsymbol{\mu}_{\text{max}} = q_i\,\boldsymbol{\mu}_F^{(i)}$$

- from fundamental representations we can construct any irreducible representation $\mathcal{D}(q_1,\ldots,q_r)$ using direct product

$$\underbrace{\mathcal{D}_F^{(1)}\otimes\ldots\otimes\mathcal{D}_F^{(1)}}_{q_1\text{-times}}\otimes\underbrace{\mathcal{D}_F^{(2)}\otimes\ldots\otimes\mathcal{D}_F^{(2)}}_{q_2\text{-times}}\otimes\ldots\otimes\underbrace{\mathcal{D}_F^{(r)}\otimes\ldots\otimes\mathcal{D}_F^{(r)}}_{q_r\text{-times}}$$

- maximal weight of the direct product is $q_i\,\boldsymbol{\mu}_F^{(i)}$

  $\Rightarrow$ the product must contain $\mathcal{D}(q_1,\ldots,q_r)$

$$\mathcal{D}_F^{(1)}\otimes\ldots\otimes\mathcal{D}_F^{(1)}\otimes\ldots\otimes\mathcal{D}_F^{(r)}\otimes\ldots\otimes\mathcal{D}_F^{(r)} = \mathcal{D}(q_1,\ldots,q_r)\oplus\text{Rest}$$

- Rest can be again decomposed by looking at the state with its maximal weight
- and so on ...

SU(3) fundamental 3 and $\bar{3}$

- 2 fundamental repr. defined

$$\boxed{\frac{2\boldsymbol{\mu}^{(i)}_{\text{max}}\cdot\boldsymbol{\alpha}_j}{|\boldsymbol{\alpha}_j|^2}=\delta_{ij},\quad \text{with } i,j=1,2}$$

$$\boldsymbol{\alpha}_1=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\quad \boldsymbol{\alpha}_2=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$$

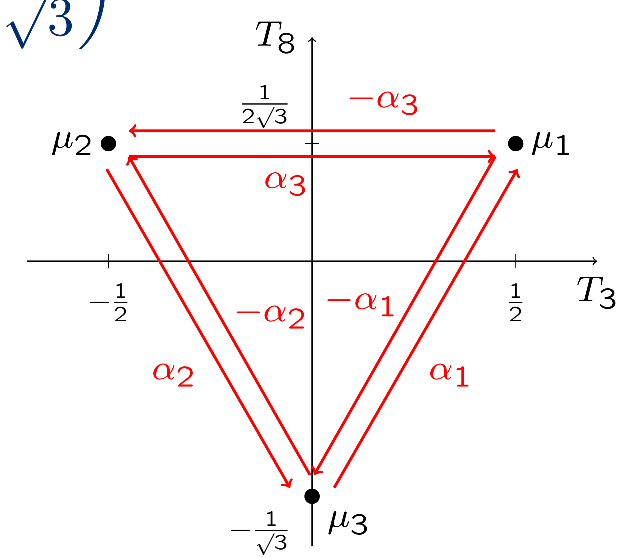
SU(3) triplet

- state with maximal $\boldsymbol{\mu}^{(1)}_{\text{max}}$ weight given by

$$2\boldsymbol{\mu}^{(1)}_{\text{max}}\cdot\boldsymbol{\alpha}_1=1\quad 2\boldsymbol{\mu}^{(1)}_{\text{max}}\cdot\boldsymbol{\alpha}_2=0$$

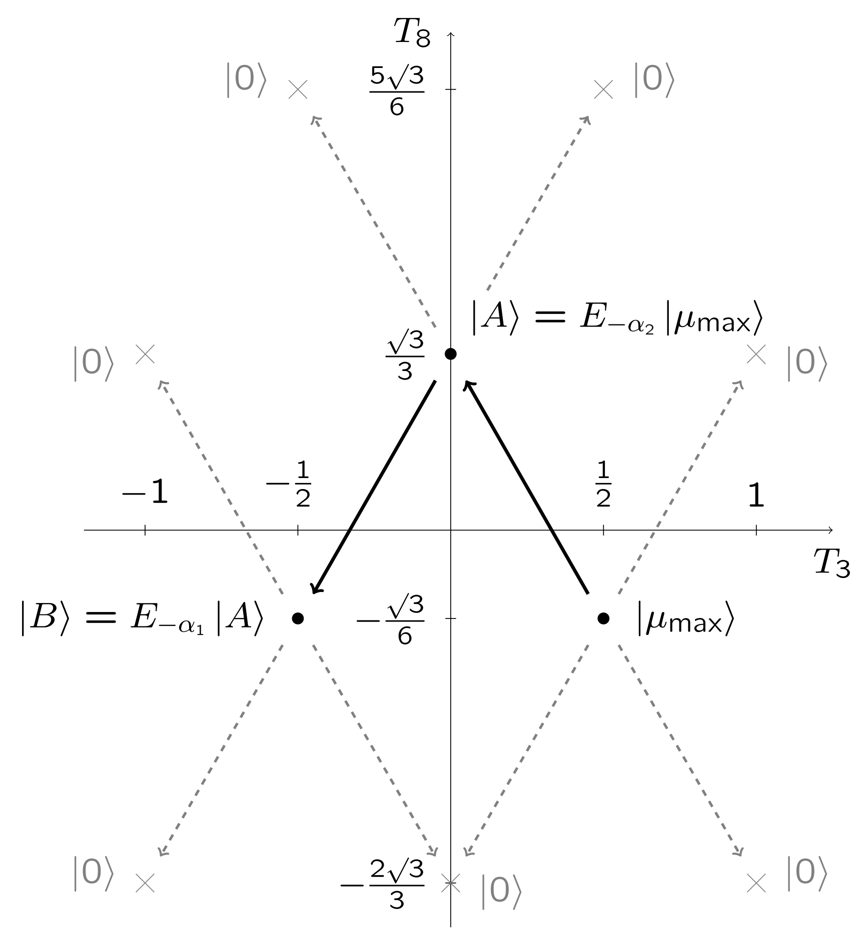
$$\Rightarrow\quad \boldsymbol{\mu}^{(1)}_{\text{max}}=\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)$$

- same state with maximal weight as in the case of matrices defining SU(3)



SU(3) anti-triplet $\bar{3}$

$$2\boldsymbol{\mu}^{(2)}_{\text{max}}\cdot\boldsymbol{\alpha}_1=0\quad 2\boldsymbol{\mu}^{(2)}_{\text{max}}\cdot\boldsymbol{\alpha}_2=1\quad\Rightarrow\quad \boldsymbol{\mu}^{(2)}_{\text{max}}=\left(\frac{1}{2},-\frac{1}{2\sqrt{3}}\right)$$



$\left|\boldsymbol{\mu}^{(2)}_{\text{max}}\right\rangle : q_1=1,\ p_1=0,\ q_2=p_2=0$

- only possible move: $|A\rangle = E_{-\boldsymbol{\alpha}_2}\left|\boldsymbol{\mu}^{(2)}_{\text{max}}\right\rangle$
- moves from $|A\rangle$, $\boldsymbol{A}=(0,\sqrt{3}/3)$
  - dir. $\boldsymbol{\alpha}_2$ clear ($p_2=1,\ q_2=0$)
  - dir. $\boldsymbol{\alpha}_1$: $2\boldsymbol{A}\cdot\boldsymbol{\alpha}_1=q_1-p_1$

    $$q_1-p_1=2\boldsymbol{A}\cdot\boldsymbol{\alpha}_1=2(0+1/2)=1$$

  - $E_{\boldsymbol{\alpha}_1}\,|A\rangle = 0$ - state with weight larger than $\left|\boldsymbol{\mu}^{(2)}_{\text{max}}\right\rangle \Rightarrow p_1=0, q_1=1$
- moves from $|B\rangle = E_{-\boldsymbol{\alpha}_1}\,|A\rangle$

$$q_2-p_2=2\boldsymbol{B}\cdot\boldsymbol{\alpha}_2=2\left(-\frac{1}{4}+\frac{1}{4}\right)=0$$

- $p_2=0$ ($E_{\boldsymbol{\alpha}_2}\,|B\rangle=0$ as $E_{-\boldsymbol{\alpha}_1}\left|\boldsymbol{\mu}^{(2)}_{\text{max}}\right\rangle=0$) $\Rightarrow q_2=0$ and we are done
- much faster using **Weyl symmetry**

Weyl symmetry and Complex conjugate repr.

Weyl symmetry

- master formula

$$\frac{2\,\boldsymbol{\alpha}\cdot\boldsymbol{\mu}}{|\boldsymbol{\alpha}|^2} = q - p$$

- implies that we can lower (if $q-p \geq 0$) $|\boldsymbol{\mu}\rangle$, or rise (if $q-p \leq 0$), at least $2\,\boldsymbol{\alpha}\cdot\boldsymbol{\mu}/|\boldsymbol{\alpha}|^2$ times (if $q-p \geq 0$)

  $\Rightarrow$ for every state $|\boldsymbol{\mu}\rangle$ there must be a state

$$\boldsymbol{\mu}_r = \mu - \frac{2\,\boldsymbol{\alpha}\cdot\boldsymbol{\mu}}{|\boldsymbol{\alpha}|^2}\boldsymbol{\alpha}$$

- in case of SU(3) and root $\boldsymbol{\alpha}_3 = (1,0)$

  $\boldsymbol{\mu} = (\mu_3, \mu_8) \;\rightarrow\; \boldsymbol{\mu}_r = (\mu_3, \mu_8) - 2\mu_3(1,0) = (-\mu_3, \mu_8)$

- it is not a chance that weights in $\bar{3}$ are minus the weights of 3: $\{\boldsymbol{\mu}_{(\bar{3})}\} = \{-\boldsymbol{\mu}_{(3)}\}$
- anti-triplet $\bar{3}$ is an example of complex conjugate repr. to triplet 3 this is the reason why it is called anti-triplet
- let's have repr. $\mathcal{D}$ on Hilbert space $\mathcal{H}$

  corresponding generators satisfy $[T_a, T_b] = i f_{abc} T_c$

Complex conjugate representation $\bar{\mathcal{D}}$

Representation on the same Hilbert space as repr. $\mathcal{D}$ with generators defined as $-(T_a^*)$

- it is easy to verify that matrices $-(T_a^*)$ satisfy the same commutator relations try it!
- relation between weights $\{\boldsymbol{\mu}_{(\bar{\mathcal{D}})}\} = \{-\boldsymbol{\mu}_{(\mathcal{D})}\}$ follows directly from hermiticity of generators

Real representation

such repr. $\mathcal{D}$ for which $\bar{\mathcal{D}}$ is equivalent with $\mathcal{D}$.

- example: in SU(2) $\bar{2} = 2$, $\binom{p}{n} \;\rightarrow\; \binom{\bar{n}}{-\bar{p}}$

Casimir operators (exercise 3.14)

- quadratic, or higher order, forms constructed out of Lie algebra of generators X_i, which commute with all its elements
- the simplest form is quadratic Casimir operator, for orthogonal generators X_i

$$C_2 = C = \sum_1^n X_i X_i \,, \qquad \forall i = 1, \dots, n: \; [C, X_i] = 0.$$

$$[X_a, X_i X_i] = X_a X_i X_i - X_i X_i X_a = [X_a, X_i] X_i + X_i X_a X_i - X_i [X_i, X_a] - X_i X_a X_i = [X_a, X_i] X_i + X_i [X_a, X_i] =$$

$$f_{aib} X_b X_i + f_{aib} X_i X_b = f_{aib}\{X_b, X_i\} = -f_{abi}\{X_b, X_i\} = -f_{abi}\{X_i, X_b\} = -f_{aib}\{X_b, X_i\} = 0 \qquad (A = -A \;\Rightarrow A = 0)$$

Schur's Lemma: Casimir operator is for irreducible representation a multiple of unit matrix $\mathbf{1}$

$\Rightarrow$ Casimir operator is not a member of algebra

- in SU(2): $C = J_1^2 + J_2^2 + J_3^2$ is square of total orbital momentum $\quad C(\mathcal{D}(j)) = j(j+1)\mathbb{1}$
- SU(3) for $\mathcal{D}(q_3, q_{\bar{3}})$: $C(q_3, q_{\bar{3}}) = \frac{1}{3}\left(q_3^2 + q_3 q_{\bar{3}} + q_{\bar{3}}^2\right) + (q_3 + q_{\bar{3}})$
- SU(N): $\quad C_F(N) = \frac{N^2-1}{2N}$, $C_A(N) = N$ (exercise 3.16)

Irreducible representation (2,0) – SU(3) sextet

- $(2,0) \Rightarrow \boldsymbol{\mu}_{\text{max}} = 2\boldsymbol{\mu}^{(1)}_{\text{max}} = (1, \sqrt{3}/3)$

$$|\boldsymbol{\mu}_{\text{max}}\rangle : q_1 = 2,\ p_1 = 0,\ q_2 = p_2 = 0$$

- 2 moves: $|A\rangle = E_{-\boldsymbol{\alpha}_1}|\boldsymbol{\mu}_{\text{max}}\rangle$, $|B\rangle = E_{-\boldsymbol{\alpha}_1}|A\rangle$
- moves from $|A\rangle$ in direction $\boldsymbol{\alpha}_2$

$$2\boldsymbol{A}\cdot\boldsymbol{\alpha}_2 = 2\left(\frac{1}{4}+\frac{1}{4}\right)1 = 1 = q_2 - p_2$$

  $q_2 = 1$ (Weyl symmetry for non-existing state $E_{-\boldsymbol{\alpha}_2}|\boldsymbol{\mu}_{\text{max}}\rangle$)

  or $p_2 = 0$ as

$$E_{\boldsymbol{\alpha}_2}|A\rangle = E_{\boldsymbol{\alpha}_2}E_{-\boldsymbol{\alpha}_1}|\boldsymbol{\mu}_{\text{max}}\rangle = [E_{\boldsymbol{\alpha}_2}, E_{-\boldsymbol{\alpha}_1}]|\boldsymbol{\mu}_{\text{max}}\rangle + E_{-\boldsymbol{\alpha}_1}E_{\boldsymbol{\alpha}_2}|\boldsymbol{\mu}_{\text{max}}\rangle = 0$$
$$= 0\ (\alpha_2 - \alpha_1 \text{ is not root})$$

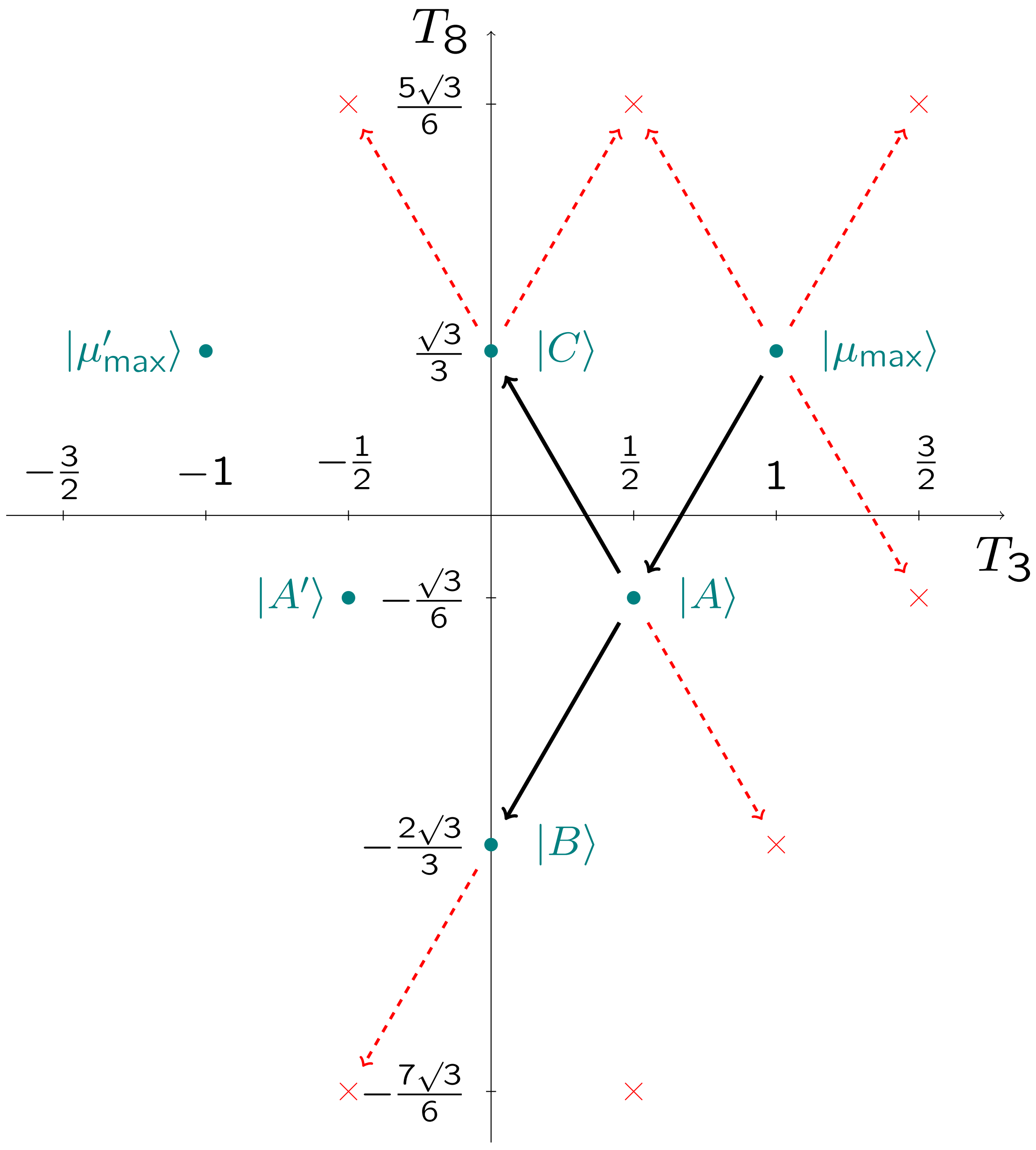
  new state $|C\rangle = E_{-\boldsymbol{\alpha}_2}|A\rangle$

- moves from $|B\rangle$ in direction $\boldsymbol{\alpha}_2$

  $E_{\boldsymbol{\alpha}_2}|B\rangle = 0$ as $E_{-\boldsymbol{\alpha}_1}|B\rangle = 0$ (Weyl sym.)

- moves from $|C\rangle$ in direction $\boldsymbol{\alpha}_1$

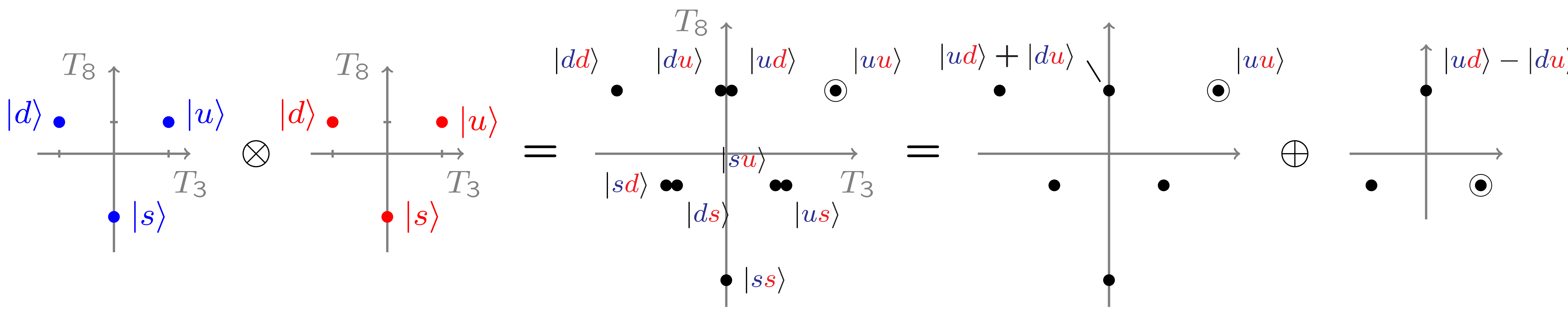
  $E_{\boldsymbol{\alpha}_1}|C\rangle = 0$ as $E_{-\boldsymbol{\alpha}_2}|\boldsymbol{\mu}_{\text{max}}\rangle = 0$ (Weyl sym.)



- we are done with $T_3 \geq 0$ half-plane
- rest is given by reflection (Weyl sym.)

Decomposition of $3 \otimes 3$

- same approach as in the case of SU(2) $2 \otimes 2$ decomposition
- first construct Hilbert space of $3 \otimes 3$
- find state with maximal weight and identify corresponding irreducible representation
- take the states from this representation out and continue until no state is left



$$\boxed{3 \otimes 3 = 6_s \oplus \bar{3}_a}$$

Irreducible representation (1,1) – SU(3) octet

- $(1,1) \Rightarrow \boldsymbol{\mu}_{\text{max}} = \boldsymbol{\mu}^{(1)}_{\text{max}} + \boldsymbol{\mu}^{(2)}_{\text{max}} = (1,0)$
- we can proceed in same way as in the case of sextet

see Exercise 3.8

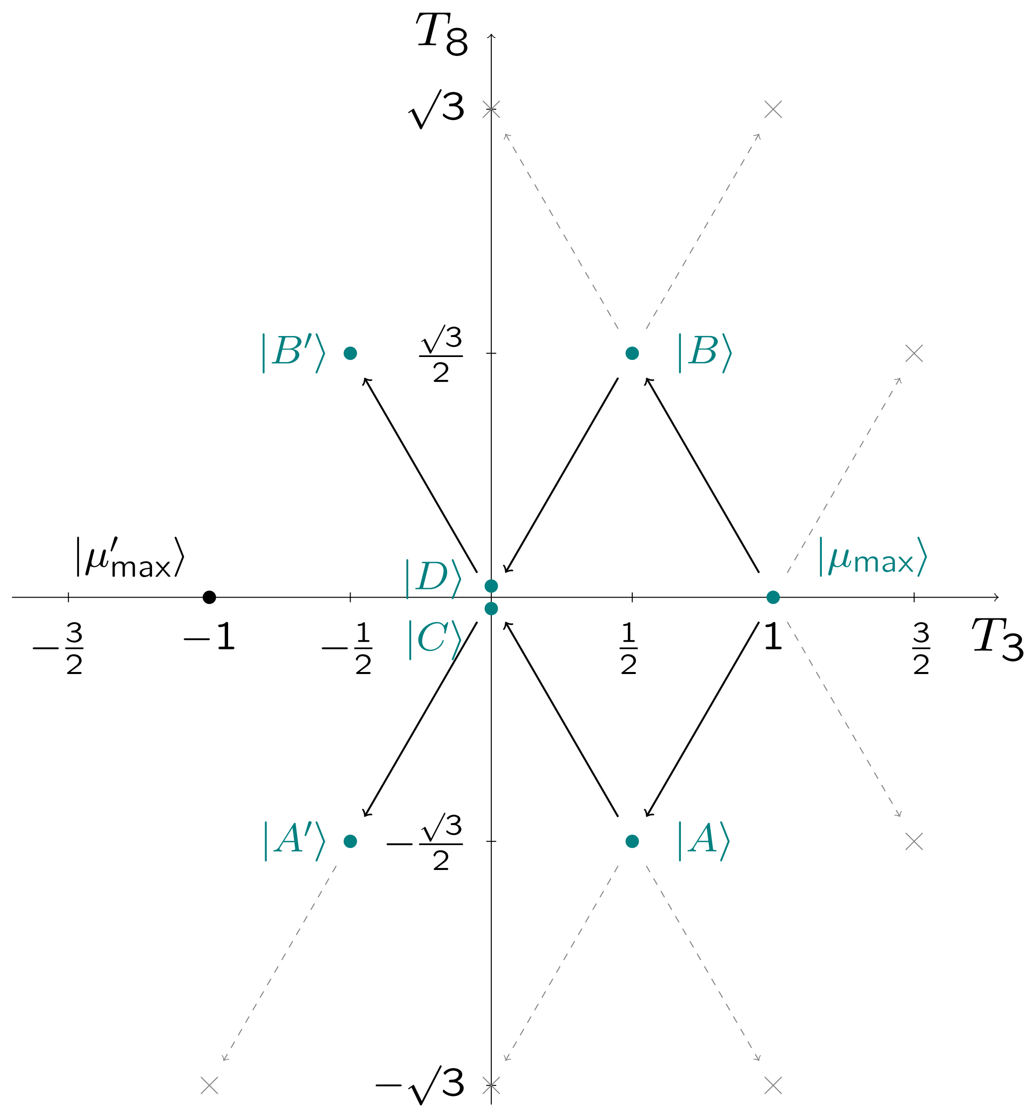
- but we know already the answer
- notice that the highest SU(3) root $\boldsymbol{\alpha}_3 = (1,0)$ is the same as $\boldsymbol{\mu}_{\text{max}}$

  $\Rightarrow$ repr. (1,1) is adjoint repr. of SU(3)

$$H_i \left| E_{\boldsymbol{\alpha}} \right\rangle = \left| [H_i, E_{\boldsymbol{\alpha}}] \right\rangle = \alpha_i \left| E_{\boldsymbol{\alpha}} \right\rangle$$

- adjoint repr. is of dimension 8 (8 generators)
  - 6 states correspond to roots
  - 2 states at $T_3 = T_8 = 0$ (they correspond to eigen-values of generators from Cartan subalgebra, which have quantum numbers equal to zero)

$\longrightarrow$

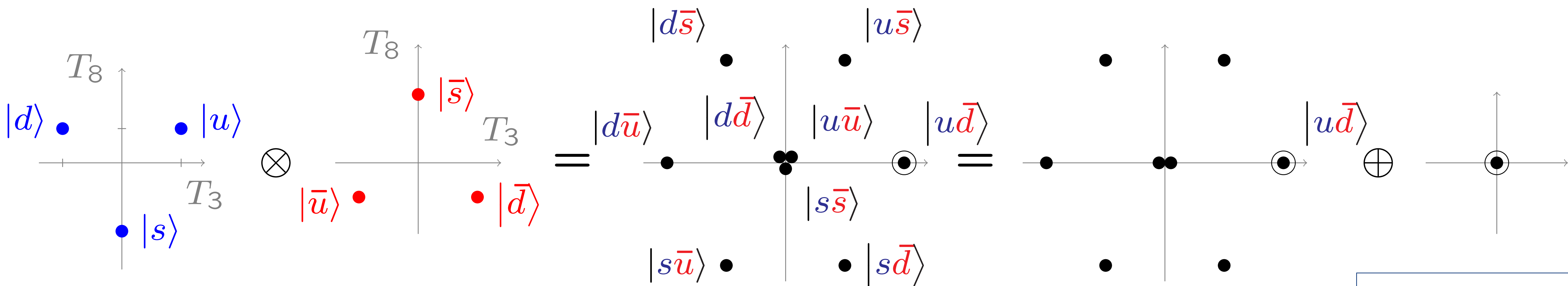


- states $|C\rangle = E_{-\boldsymbol{\alpha}_2} E_{-\boldsymbol{\alpha}_1} \left| \boldsymbol{\mu}_{\text{max}} \right\rangle$ and $|D\rangle = E_{-\boldsymbol{\alpha}_1} E_{-\boldsymbol{\alpha}_2} \left| \boldsymbol{\mu}_{\text{max}} \right\rangle$ linearly independent

$$\frac{\langle C \,|\, D \rangle \langle D \,|\, C \rangle}{\langle C \,|\, C \rangle \langle D \,|\, D \rangle} \neq 1$$

Decomposition of $3 \otimes \bar{3}$

Exercise 3.11



$$3 \otimes \bar{3} = 8 \oplus 1$$

- phase of normalization factor $N_{\boldsymbol{\alpha},\boldsymbol{\mu}}$ not fixed

$$|N_{-\boldsymbol{\alpha},\boldsymbol{\mu}}|^2 = \frac{(p+1)q}{2} |\boldsymbol{\alpha}|^2$$

- once we made a choice for 3, phase of shifting operators in $\bar{3}$ is fixed

$$E_{-\boldsymbol{\alpha}_1} |u\rangle = \frac{1}{\sqrt{2}} |d\rangle \qquad E_{-\boldsymbol{\alpha}_2} |\bar{d}\rangle = \frac{1}{\sqrt{2}} e^{i\varphi} |\bar{s}\rangle$$

Singlet wave function: $|1\rangle = a\,|u\bar{u}\rangle + b\,|d\bar{d}\rangle + c\,|s\bar{s}\rangle$

$$E_{-\boldsymbol{\alpha}_3} |1\rangle \propto a\,|d\bar{u}\rangle - b\,|d\bar{u}\rangle = 0 \;\Rightarrow\; a = b$$

$$E_{-\boldsymbol{\alpha}_1} |1\rangle \propto a\,|s\bar{u}\rangle - c\,|d\bar{d}\rangle = 0 \;\Rightarrow\; a = c (= b)$$

$$|1\rangle = \frac{1}{\sqrt{3}} \left(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle \right)$$

$$|C\rangle = E_{-\boldsymbol{\alpha}_2} E_{-\boldsymbol{\alpha}_1} |u\bar{d}\rangle = \frac{1}{\sqrt{2}} E_{-\boldsymbol{\alpha}_2} |s\bar{d}\rangle = \frac{1}{2} \left(|d\bar{d}\rangle + e^{i\varphi} |s\bar{s}\rangle \right)$$

$$|D\rangle = E_{-\boldsymbol{\alpha}_1} E_{-\boldsymbol{\alpha}_2} |u\bar{d}\rangle = \frac{e^{i\varphi}}{\sqrt{2}} E_{-\boldsymbol{\alpha}_1} |u\bar{s}\rangle = \frac{1}{2} \left(e^{i\varphi} |s\bar{s}\rangle + e^{2i\varphi} |u\bar{u}\rangle \right)$$

- $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3 \Rightarrow [E_{-\boldsymbol{\alpha}_1}, E_{-\boldsymbol{\alpha}_2}] \propto E_{-\boldsymbol{\alpha}_3}$

$$[E_{-\boldsymbol{\alpha}_1}, E_{-\boldsymbol{\alpha}_2}] |u\bar{d}\rangle = |D\rangle - |C\rangle = \frac{1}{2} \left(e^{2i\varphi} |u\bar{u}\rangle - |d\bar{d}\rangle \right)$$

$$\propto E_{-\boldsymbol{\alpha}_3} |u\bar{d}\rangle = \frac{1}{\sqrt{2}} (|d\bar{d}\rangle + e^{i\varphi} |u\bar{u}\rangle)$$

$$\Rightarrow e^{2i\varphi} = -e^{i\varphi} \Rightarrow e^{i\varphi} = -1$$ expected from SU(2)

Exercises

- in addition to exercises mentioned in the lecture
  - Construct SU(3) irreducible repr. $(1, 1)$ - octet, using shifting operators Exercise 3.8
  - Construct SU(3) irreducible repr. $(3, 0)$ - decuplet Exercise 3.9
  - for fun - higher SU(3) multiplets Exercise 3.13
  - Construct SU(4) fundamental representations Exercise 3.15
  - Generalization to SU(N) Exercise 3.16
- plan for today's exercises
  - Construct SU(3) irreducible repr. $(3, 0)$ - decuplet Exercise 3.9
  - Decomposition $3 \otimes 3 \otimes 3$ Exercise 3.12
  - Casimir operators of SU(2) and SU(3) Exercise 3.14