Quarks, Partons, and Quantum Chromodynamics

Lecture 2

Groups and symmetries - short introduction

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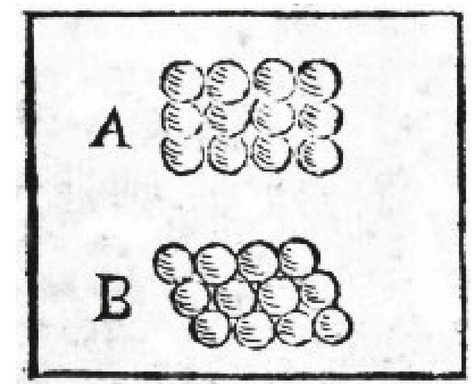
Recommended literature:
H. Georgi, Lie Algebras in Particle Physics

Symmetries in physics

- symmetries play an important role in physics
- symmetry is a reflection of simpler internal structure
- symmetries are related to **conservation laws**



Kepler 1611




Bentley 1902

Noether's theorem

Every differentiable symmetry of the action ($S = \int \mathcal{L} \mathrm{d}t$) has a corresponding conservation law.

translation in space $\rightarrow$ momentum conservation, time $\rightarrow$ energy, rotation $\rightarrow$ orbital momentum, ...

- in QFT, symmetries are related with the dynamics as well (!)
  - electro-magnetic, electro-weak and strong interactions are all QFT with local gauge invariance
  - local invariance with respect to the internal (charge related) transformations:

group $U(1)$ for QED, $SU(2) \times U(1)$ for EWK, $SU(3)$ for QCD

- **Groups** - mathematical tool to classify the symmetries and work with them
  - Évariste Galois introduced the term group and used them in 1830 to determine the solvability of polynomial equations

Groups - basic concepts

Group - set G with binary operation "$\cdot$"

1) G is closed under "$\cdot$"

   $\forall\, x, y \in G:\ x \cdot y \in G$

2) unit element

   $\exists\, e \in G:\ \forall\, x \in G:\ x \cdot e = e \cdot x = x$

3) inverse element

   $\forall\, x \in G:\ \exists\, x^{-1} \in G:\ x \cdot x^{-1} = x^{-1} \cdot x = e$

4) associativity

   $\forall\, x, y, z \in G:\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Examples

- real numbers $\mathbb{R}$ with operation "$+$"
- translations in 3D space
- rotations in 3D space (try to prove associativity)

Group representation - $\mathcal{D}$

Mapping $\mathcal{D}$ from G to a set of linear operators $\mathcal{L}_{\mathcal{H}}$ on Hilbert space $\mathcal{H}$ ($g \in G \to \mathcal{D}(g) \in \mathcal{L}_{\mathcal{H}}$) that preserves operation "$\cdot$"

$$\forall\, x, y, z \in G : x \cdot y = z \;\Rightarrow\; \mathcal{D}(x)\mathcal{D}(y) = \mathcal{D}(z)$$

- very useful concept in physics
- **Hilbert space** represents in QM the space of states (vector space equipped with an inner product that allows to define distances and angles)
- we will always use the matrix representation of linear operators

$$\mathcal{O}_{ij} \equiv \langle i|\, \mathcal{O}\, |j\rangle \quad \Rightarrow \quad \mathcal{O}\, |j\rangle = \mathcal{O}_{ij}\, |i\rangle \qquad \text{(sum notation)}$$

  where $|i\rangle$, $|j\rangle$ run over base vectors of $\mathcal{H}$

  - for real numbers with op. "$+$": $x \in \mathbb{R} : \mathcal{D}(x) = \exp(ix)$ on 1-dim $\mathcal{H}$
  - rotations in 3D space repr. by 3×3 matrices R: $\vec{x} \to \vec{x}\,' = R\,\vec{x}$, SO(3)

- group G defines the set of operators (symmetry) under which our system (described by $\hat{H}$) behaves the same

Groups - basic concepts

- values of $\mathcal{O}_{ij}$ depend on choice of the basis of $\mathcal{H}$

Two representations $\mathcal{D}_1$ and $\mathcal{D}_2$ are equivalent if

$$\exists \text{ unitary } S \in \mathcal{L}_\mathcal{H}: \ \forall x \in G: \ \mathcal{D}_1(x) = S\mathcal{D}_2(x)S^{-1}$$

- two representations $\mathcal{D}_1$ and $\mathcal{D}_2$

  $\mathcal{D}_1$ acts on $\mathcal{H}_1$ with base $|e_1^1\rangle, |e_2^1\rangle, \ldots, |e_{n_1}^1\rangle$
  $\mathcal{D}_2$ acts on $\mathcal{H}_2$ with base $|e_1^2\rangle, |e_2^2\rangle, \ldots, |e_{n_2}^2\rangle$

- several ways how to combine them into a new one
  - direct sum describes two separate worlds
  - direct product describes two interacting systems
- repr. $\mathcal{D}$ is reducible if

$$\exists \text{ unitary } S \in \mathcal{L}_\mathcal{H}: \ \forall x \in G: \ S\mathcal{D}(x)S^{-1} = \begin{pmatrix} \mathcal{D}_1(x) & 0 \\ 0 & \mathcal{D}_2(x) \end{pmatrix}$$

  such $\mathcal{D}$ can be then written as direct sum $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$
- if unitary S does not exists, representation is irreducible

Direct sum - $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$

- $\mathcal{D}$ acts on Hilbert space which is a direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, of dimension $n_1 + n_2$ with basis $|e_1^1\rangle, |e_2^1\rangle, \ldots, |e_{n_1}^1\rangle, |e_1^2\rangle, |e_2^2\rangle, \ldots, |e_{n_2}^2\rangle$
- group element $x \in G$ is mapped to block matrix $(n_1 + n_2) \times (n_1 + n_2)$

$$\mathcal{D}(x) = \begin{pmatrix} \mathcal{D}_1(x) & 0 \\ 0 & \mathcal{D}_2(x) \end{pmatrix}$$

Direct product - $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$

- $\mathcal{D}$ acts on $n_1 \times n_2$ dimensional $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$|e_1^1\rangle |e_1^2\rangle, |e_2^1\rangle |e_1^2\rangle, \ldots, |e_{n_1}^1\rangle |e_1^2\rangle, \ldots$$
$$\ldots, |e_1^1\rangle |e_{n_2}^2\rangle, |e_2^1\rangle |e_{n_2}^2\rangle, \ldots, |e_{n_1}^1\rangle |e_{n_2}^2\rangle$$

- $\mathcal{D}(x)$ is $n_1 \times n_2$ dimensional square matrix

$$\mathcal{D}_1(x) \otimes \mathcal{D}_2(x) |e_i^1\rangle |e_j^2\rangle = \mathcal{D}_1(x) |e_i^1\rangle \mathcal{D}_2(x) |e_j^2\rangle$$

$$[(\mathcal{D}_1 \otimes \mathcal{D}_2)(x)]_{ij,i'j'} = [\mathcal{D}_1(x)]_{ii'} [\mathcal{D}_2(x)]_{jj'}$$

Symmetries and conservations in QM - rotational invariance

Experiment does not depend on rotation of coordinates

$$R: \; \vec{x} \; \rightarrow \; \vec{x}\,' = R\,\vec{x}$$

$$\text{in QM}: \; |\psi\rangle \; \rightarrow \; |\psi'\rangle = \hat{U}(R)\,|\psi\rangle$$

- independence on choice of coordinates implies

$$\forall\, |\varphi\rangle\,, |\psi\rangle \in \mathcal{H}:$$

$$|\langle\varphi|\,\psi\rangle|^2 = |\langle\varphi'|\,\psi'\rangle|^2 = |\langle\varphi|\,\hat{U}^+(R)\hat{U}(R)\,|\psi\rangle|^2$$

- only possible (without proof) if

$$\hat{U}^+(R)\hat{U}(R) = \mathbb{1}$$

- relevant operators on $\mathcal{H}$ describing symmetries are **unitary**

Let's have a physical system (described by Hamiltonian $\hat{H}$) with (rotational) symmetry

$$|\langle\varphi|\,\hat{H}\,|\psi\rangle|^2 = |\langle\varphi'|\,\hat{H}\,|\psi'\rangle|^2 = |\langle\varphi|\,\hat{U}^+(R)\hat{H}\hat{U}(R)\,|\psi\rangle|^2$$

$$\hat{H} = \hat{U}^+\hat{H}\hat{U} \qquad \Rightarrow \qquad [\hat{H},\hat{U}] = 0$$

- $\hat{H}$ commutes with all operators representing the symmetry group

$$i\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|\,\hat{U}\,|\psi(t)\rangle = \left\langle i\frac{\mathrm{d}\psi}{\mathrm{d}t}\middle|\hat{U}\middle|\psi\right\rangle + \left\langle\psi\middle|\hat{U}\middle|i\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle = \langle\psi|-\hat{H}\hat{U}+\hat{U}\hat{H}\,|\psi\rangle = 0$$

Schrödinger eq.: $\quad -\langle\psi(t)|\,\hat{H} \qquad\qquad \hat{H}\,|\psi(t)\rangle$

- we found (many) **integrals of motion**
- to get corresponding conserved observable we need Hermitian operator – notion of **generators**

Rotational invariance

$\boxed{\text{Consider system invariant with respect to rotations along some (the third) axis}}$

- consider infinitesimally small rotation at angle $\varepsilon \to 0$

$$\hat{U}(\varepsilon) = 1 - i\varepsilon \hat{J}_3$$

- unitarity of $\hat{U}$ leads to the condition on $\hat{J}_3$

$$\mathbb{1} = \hat{U}^{+}\hat{U} = (1 + i\varepsilon \hat{J}_3^{+})(1 - i\varepsilon \hat{J}_3) = 1 + i\varepsilon(\hat{J}_3^{+} - \hat{J}_3) + \mathcal{O}(\varepsilon^2)$$

$$\boxed{\hat{J}_3^{+} = \hat{J}_3}$$

- we found Hermitian operator which mean value remains constant over time
- and we have physical observable that is conserved

compare with Noether's theorem

$\boxed{\text{What is the conserved quantity related to the symmetry with respect to the rotation along the third axis?}}$

$$\vec{r}\,' = R(\varepsilon)\vec{r}: \quad x' = x + \varepsilon y, \quad y' = -\varepsilon x + y, \quad z' = z \qquad R(\varepsilon) = \begin{pmatrix} 1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ○ in space representation $\hat{p}_x = -i\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial x} = i\hat{p}_x$

$$U(\varepsilon)\psi(\vec{r}) = \psi(R(\varepsilon)\vec{r}) = \psi(x + \varepsilon y, -\varepsilon x + y, z) = \psi(\vec{r}) + \varepsilon\left[\frac{\partial \psi}{\partial x}y - \frac{\partial \psi}{\partial y}x\right] = (\mathbb{1} - i\varepsilon\left[x\hat{p}_y - y\hat{p}_x\right])\,\psi(\vec{r})$$

$$\boxed{\hat{J}_3 = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x}$$

- conserving physics quantity is **orbital momentum**
- it is not difficult to prove

$$\left[\hat{J}_i, \hat{J}_j\right] = i\varepsilon_{ijk}\hat{J}_k$$

Lie Groups

- group of rotations is an example of **Lie group**
- rotation can be characterized by 3 real numbers
  (Euler's angles, direction of axis of rotation + angle, ...)
  and the map from G to $\mathbb{R}^3$ is continuous

Lie group

Such group G, for which there exists a local continuous map from G to $\mathbb{R}^n$, $G \to \mathbb{R}^n$ such that

$$x, y \in G : x \cdot y \to \mathbb{R}^n$$

is continuous in x and y and x^{-1} is continuous in x.

- note that we can construct any rotation from infinite sequence of infinitesimal rotations

$$\hat{U}(\vartheta) = \lim_{n\to\infty} \left[\hat{U}\left(\frac{\vartheta}{n}\right)\right]^n = \lim_{n\to\infty} \left(1 - i\frac{\vartheta}{n}\hat{J}_3\right)^n = \exp\left(-i\vartheta\hat{J}_3\right)$$

Proposition 1: Any element of compact Lie group G can be written in the form

$$\forall g \in G : \; g = g(\alpha_a) = \exp(i\alpha_a \hat{X}_a), \quad \alpha_1, \ldots, \alpha_m \in \mathbb{R}.$$

- operators $X_1, \ldots, X_m$ are called generators ($\hat{}$ skipped)
- they form basis of vector space (over the field of $\mathbb{C}$) of dimension m with operator addition "+"

Proposition 2: All irreducible representations of compact Lie groups are of finite dimension.

Proposition 3: Finite dimensional representations of a compact Lie group are equivalent to representation by unitary operators.

$\Rightarrow$ generators X_a are Hermitian operators

Lie algebras

C-algebra - $\mathcal{A}$

Vector space on field of $\mathbb{C}$ with additional bilinear form "$\times$" satisfying $\forall\, x, y, z \in \mathcal{A}$ and $\forall\, a, b \in \mathbb{C}$:

- form "$\times$" is closed on $\mathcal{A}$: $x \times y \in \mathcal{A}$
- distributivity
$$(x+y) \times z = x \times z + y \times z$$
$$x \times (y+z) = x \times y + x \times z$$
- bilinearity
$$(ax) \times (by) = (ab)(x \times y)$$

Lie algebra

- vector space formed over base made out of generators X_a
$$|X_1\rangle\,, |X_2\rangle\; \ldots, |X_m\rangle$$
- role of bilinear form is played by commutator

- let's prove that commutator is closed on $\mathcal{A}$:

  Proposition 1 implies that
$$e^{i\lambda X_a} e^{i\lambda X_b} e^{-i\lambda X_a} e^{-i\lambda X_b} = e^{i\alpha_c X_c}, \text{ with } \alpha_c, \lambda \in \mathbb{R}$$
  Taylor expansion for $\lambda \to 0$
$$\left(\mathbb{1} + i\lambda X_a - \lambda^2 X_a^2/2\right)\left(\mathbb{1} + i\lambda X_b - \lambda^2 X_b^2/2\right)\left(\mathbb{1} - i\lambda X_a - \lambda^2 X_a^2/2\right)$$
$$\left(\mathbb{1} - i\lambda X_b - \lambda^2 X_b^2/2\right) = \mathbb{1} + i\lambda\left(X_a + X_b - X_a - X_b\right)$$
$$-\lambda^2\left(X_a^2 + X_b^2 + X_a X_b - X_a^2 - X_a X_b - X_b X_a - X_b^2 + X_a X_b\right)$$
$$= \mathbb{1} - \lambda^2\left[X_a,\, X_b\right] = \mathbb{1} + i\alpha_c X_c$$
$$\boxed{[X_a,\, X_b] = i f_{abc} X_c}$$

- $f_{abc} \in \mathbb{R}$ ... **structure constants**
  - determine group algebra
  - two different groups can have same f_{abc}: $SU(2)$ and $SO(3)$

    however $SU(2) \neq SO(3)$, $SU(2)$ can be mapped to two SO(3)
  - Jacobi identity for commutators implies   try it!
$$f_{ade} f_{bcd} + f_{cde} f_{abd} + f_{bde} f_{cad} = 0$$

Representation of Lie Algebras

Representation of Lie Algebra

Mapping $\mathcal{D}$ from algebra's vector space $\mathcal{A}$ to $\mathcal{L}_{\mathcal{H}}$ that preserves the commutator

$$\forall\, x, y, z \in \mathcal{A} : [x, y] = z \;\Rightarrow\; [\mathcal{D}(x), \mathcal{D}(x)] = \mathcal{D}(z)$$

Generators $X_a^{\mathcal{D}_1 \otimes \mathcal{D}_2}$ of $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$

$$g \in G \;\to \vec{\alpha} : \quad \begin{aligned} \mathcal{D}_1(g) &= \exp(i\alpha_a X_a^{\mathcal{D}_1}) \\ \mathcal{D}_2(g) &= \exp(i\alpha_a X_a^{\mathcal{D}_2}) \\ \mathcal{D}_1 \otimes \mathcal{D}_2(g) &= \exp(i\alpha_a X_a^{\mathcal{D}_1 \otimes \mathcal{D}_2}) \end{aligned}$$

- in representation with matrices

$$\begin{aligned} \exp(i\alpha_a X_a^{\mathcal{D}_1 \otimes \mathcal{D}_2})_{ij,i'j'} &= \exp(i\alpha_a X_a^{\mathcal{D}_1})_{ii'} \exp(i\alpha_a X_a^{\mathcal{D}_2})_{jj'} \\ &= \exp\left[i\alpha_a (X_a^{\mathcal{D}_1})_{ii'} \mathbb{1}_{jj'} + \mathbb{1}_{ii'} i\alpha_a (X_a^{\mathcal{D}_2})_{jj'} \right] \end{aligned}$$

$$\boxed{(X_a^{\mathcal{D}_1 \otimes \mathcal{D}_2})_{ij,i'j'} = (X_a^{\mathcal{D}_1})_{ii'} \mathbb{1}_{jj'} + \mathbb{1}_{ii'} (X_a^{\mathcal{D}_2})_{jj'}}$$

rule for the composition of orbital momenta of 2 particles

Choice of base of Lie algebra

- $\text{Tr}(X_a X_b^+)$ is positive real symmetric tensor (prove it)

  show that $[\text{Tr}(X_a X_b)]^\star = \text{Tr}(X_a X_b)$ and $\text{Tr}(X_a X_a) > 0$

  $\Rightarrow$ we can always find orthogonal base

$$\text{Tr}(X_a X_b^+) = \text{Tr}(X_a X_b) = \lambda \delta_{ab}, \quad \lambda \in \mathbb{R},\ \lambda > 0$$

- conventional choice $\lambda = 1/2$

- with this choice we have

$$\text{Tr}([X_a, X_b] X_c) = i f_{abd} \text{Tr}(X_d X_c) = i\lambda f_{abc}$$

- however

$$\begin{aligned} \text{Tr}([X_a, X_b] X_c) &= \text{Tr}([X_c, X_a] X_b) = \text{Tr}([X_b, X_c] X_a) \\ &= \text{Tr}(X_a X_b X_c - X_b X_a X_c) = \text{Tr}(X_c X_a X_b - X_a X_c X_b) \end{aligned}$$

$$\Rightarrow f_{abc} = f_{cab} = f_{bca}$$

- from definition $f_{abc} = -f_{bac}$

  $\Rightarrow$ structure constants f_{abc} are **fully antisymmetric**

Adjoint representation

- special representation constructed from linear operators on vector space of algebra with base

$$|X_1\rangle\,,|X_2\rangle\,,\ldots,|X_m\rangle$$

- we need Hilbert space - vector space with scalar product, which we can introduce as

$$\langle X_b|\,X_a\rangle \equiv \frac{1}{\lambda}\mathrm{Tr}(X_b^+ X_a)\ (=\delta_{ab})$$

- it is straightforward to show that m square $m\times m$ matrices $(F_a)_{bc} = -if_{abc}$ satisfy

$$[F_a, F_b] = if_{abc}F_c$$

  and they represent generators of representation called **adjoint** try it, you will need Jacobi identity
- very special role in Yang-Mills QFT - particles mediating the force (photons, gluons) are represented by adjoint repr.
- also very handful in mathematical constructions

$$|F_1\rangle = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\ |F_2\rangle = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\ \ldots,\ |F_m\rangle = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix} \quad\Rightarrow\quad F_a\,|F_b\rangle = (F_a)_{1b}\,|F_1\rangle + \ldots + (F_a)_{mb}\,|F_m\rangle = -if_{acb}\,|F_c\rangle = |if_{abc}F_c\rangle = |[F_a, F_b]\rangle$$

- linearity $\Rightarrow$ in adjoint representation: $\boxed{\hat{A}\,|B\rangle = |[A,B]\rangle}$

Weights, roots and shifting operators

- Lie group with m generators with some representation $\mathcal{D}$
- we can characterize the eigenstates of $\mathcal{H}$ by corresponding eigenvalues (quantum numbers)

> **Rank** (r) – maximal number of commuting generators
>
> $$\forall\, i,j = 1,\ldots,r:\ [H_i, H_j] = 0$$

- ◦ vector space from base $|H_1\rangle, \ldots, |H_r\rangle$ forms **Cartan subaglebra**
- base of $\mathcal{H}$ can be chosen as common eigenstates of H_i

$$H_i\,|\vec{\mu}, \mathcal{D}\rangle = \mu_i\,|\vec{\mu}, \mathcal{D}\rangle$$

  where $\vec{\mu} = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$ are called **weights**

- remaining $m-r$ generators can be organized to **shifting operators** $E_{\vec{\alpha}}$: $\boxed{[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}}$ $\quad \langle H_i|H_j\rangle = \delta_{ij}$, $\quad \langle E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = \delta_{\vec{\alpha}\vec{\beta}}$

$$H_i E_{\vec{\alpha}}\,|\vec{\mu}\rangle = [H_i, E_{\vec{\alpha}}]\,|\vec{\mu}\rangle + E_{\vec{\alpha}} H_i\,|\vec{\mu}\rangle = (\mu_i + \alpha_i) E_{\vec{\alpha}}\,|\vec{\mu}\rangle \Rightarrow E_{\vec{\alpha}}\,|\vec{\mu}\rangle \propto |\vec{\mu} + \vec{\alpha}\rangle$$

- $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ ... **roots** (do not depend on $\mathcal{D}$)

- in adjoint representation: $H_i\,|E_{\vec{\alpha}}\rangle = |[H_i, E_{\vec{\alpha}}]\rangle = \alpha_i\,|E_{\vec{\alpha}}\rangle$
  - $\Rightarrow$ roots are weights of adjoint repr. ($\alpha_i \in \mathbb{R}$)
  - $\Rightarrow$ $E_{\vec{\alpha}}$ are eigenstates of H_i in adjoint repr.
  - $\Rightarrow$ recipe how to prove the existence of $E_{\vec{\alpha}}$
  - $\Rightarrow$ and how to construct them explicitly
- H_i has in adj. repr. r rows and columns with zeros

$$H_i = \begin{array}{cc} \begin{array}{cc} \text{r} & \text{m-r} \end{array} & \\ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (H_i)_{ab} = -i f_{iab} \end{array}\right) & \begin{array}{c} \text{r} \\ \text{m-r} \end{array} \end{array}$$

- remaining $m-r$ square matrix is linear indep.

  if not there would be another generator commuting with all H_i

  $\Rightarrow$ there must be $m-r$ eigenstates: $E_{\vec{\alpha}_1}, \ldots, E_{\vec{\alpha}_{m-r}}$
- some useful properties

  $E^{+}_{\vec{\alpha}} = E_{-\vec{\alpha}}, \quad [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \alpha_i H_i$ (prove them!)

SU(2) - example 2.1

$SU(2)$ group – a set of 2×2 unitary (U) complex matrices with unit determinant (S - special)

- 2×2 matrices form also representation on 2-dim $\mathcal{H}$
- generators are 2×2 Hermitian traceless matrices

$$\det\left(e^{A}\right) = e^{\mathrm{Tr}(A)}$$

- traceless Hermitian matrix can be parameterized with 3 real numbers a, b, and c

$$\begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}$$

- $SU(2)$ is Lie group with 3 generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  well known Pauli matrices, check that they satisfy

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k \quad \Rightarrow \quad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$\Rightarrow \quad \mathrm{Tr}(\sigma_i\sigma_j^{+}) = \mathrm{Tr}(\mathbb{1})\delta_{ij} = 2\delta_{ij}$$

- conventional norm for $J_i = \frac{1}{2}\sigma_i$: $[J_i, J_j] = i\varepsilon_{ijk}J_k$

- SU(2) rank is $r=1$: $H_1 \equiv J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
- 2 eigenstates forming basis of $\mathcal{H}$: $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- corresponding weights: $\vec{\mu_1} = (\frac{1}{2})$ and $\vec{\mu_2} = (-\frac{1}{2})$
- $3-1=2$ roots: $\vec{\alpha_1} = \vec{\mu_1} - \vec{\mu_2} = (1)$, $\vec{\alpha_2} = -\vec{\alpha_1} = (-1)$
- construction of shifting operators in adjoint repr.

  J_3 in adjoint repr. reads

$$F_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  eigenstates (shifting op.) and eigenvalues (roots) are solutions of

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} x \\ y \end{pmatrix} \;\Rightarrow\; -iy = \alpha x,\; ix = \alpha y \;\Rightarrow\; y = \alpha^2 y$$

  roots are $\alpha_{1,2} = \pm 1$ and corresponding eigenstates are

$$|E_{\alpha=1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \; |E_{\alpha=-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \langle E_\alpha | E_\beta\rangle = \delta_{\alpha\beta}$$

$$\Rightarrow J_+ \equiv E_1 = \frac{J_1 + iJ_2}{\sqrt{2}}, \quad J_- \equiv E_{-1} = \frac{J_1 - iJ_2}{\sqrt{2}}, \quad [J_+, J_-] = J_3$$

SU(2) irreducible repr.

- Prop. 2 $\Rightarrow$ SU(2) irred. rep. are of finite dim.

$\exists\, |j\rangle$ with max weight j: $J_3 |j\rangle = j\,|j\rangle\,,\ J^+ |j\rangle = 0$

- starting point for construction of $\mathcal{D}(j)$ using J^-
- J_3 forms Cartan subalgebra

$$\mathcal{H}\text{ basis}:\ \ J_3 |j,m\rangle = m\,|j,m\rangle$$

- $k = j - m$ counts how many times we lowered $|j\rangle$
- norm of J^-: $J^- |j, m = j-k\rangle = N_k\,|j, m-1\rangle$

$$\Rightarrow\ N_k = \langle j-k-1|\, J^-\, |j-k\rangle\,.$$

- complex conj. gives $N_k^\star = \langle j-k|\, J^+\, |j-k-1\rangle$

$$\Rightarrow\ J^+ |m = j-k-1\rangle = N_k^\star\,|m = j-k\rangle$$

$$N_k^\star N_k = \langle m|\, J^+ J^-\, |m\rangle = \langle m|\, \underbrace{[J^+, J^-]}_{J_3}\, |m\rangle + \langle m|\, J^- \underbrace{J^+ |m\rangle}_{N_{k-1}\,|m+1\rangle}$$

$$= \langle m|\, J_3\, |m\rangle + N_{k-1}^\star N_{k-1} = j - k + N_{k-1}^\star N_{k-1}$$

$$j - k = |N_k|^2 - |N_{k-1}|^2$$

$$\begin{array}{lll}
k=0 \bullet & |j\rangle: & j = |N_0|^2 \\
 & \downarrow J^- & \\
k=1 \bullet & |j-1\rangle: & j-1 = |N_1|^2 - |N_0|^2 \\
 & \downarrow J^- & \\
k=2 \bullet & |j-2\rangle: & j-2 = |N_2|^2 - |N_1|^2 \\
 & \vdots & \\
k \bullet & |j-k\rangle: & j-k = |N_k|^2 - |N_{k-1}|^2 \\
\hline
\end{array}$$

$$\sum:\ \frac{(k+1)(2j-k)}{2} = |N_k|^2$$

- finite size of irred. repr. $\Rightarrow$ max weight j is half-integer

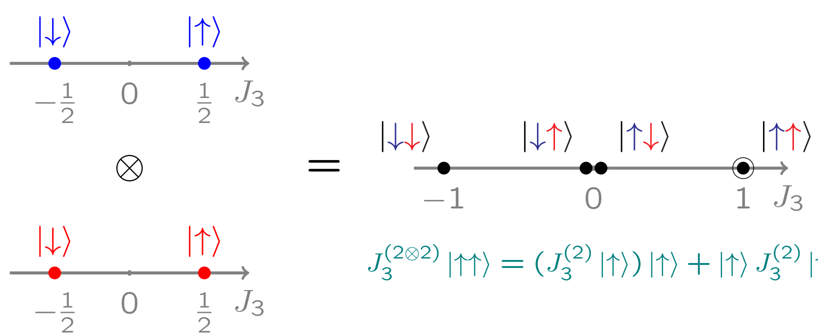
$$N_{2j} = 0$$

$\Rightarrow$ irreducible repr. $\mathcal{D}(j)$ has $2j+1$ dimensional $\mathcal{H}$

$$|-j\rangle\,,\ |-j+1\rangle\ \ \ldots,\ |j-1\rangle\,,\ |j\rangle$$

SU(2): $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$

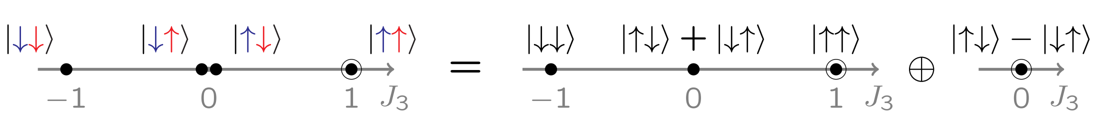
- $\mathbf{2}$ - SU(2) irr. repr. with $\dim_{\mathcal{H}} = 2$: $\mathcal{D}(j = 1/2)$
  - this is also fundamental repr. of SU(2)
  - $\mathcal{H}$ represent spin states of particle with spin $1/2$
  - doublet $|\uparrow\rangle$, $|\downarrow\rangle$: $J_3\,|\uparrow\rangle = \frac{1}{2}\,|\uparrow\rangle\,,\quad J_3\,|\downarrow\rangle = -\frac{1}{2}\,|\downarrow\rangle$
- Hilbert space of two fermions with spin $1/2$: $\mathbf{2} \otimes \mathbf{2}$



$J_3^{(2\otimes 2)}\,|\uparrow\uparrow\rangle = (J_3^{(2)}\,|\uparrow\rangle)\,|\uparrow\rangle + |\uparrow\rangle\,J_3^{(2)}\,|\uparrow\rangle = (\tfrac{1}{2} + \tfrac{1}{2})\,|\uparrow\rangle\uparrow$

- find state with max weight $j = 1$
- identify corresponding irr. repr (triplet $\mathbf{3}$)
- and continue until no state is left

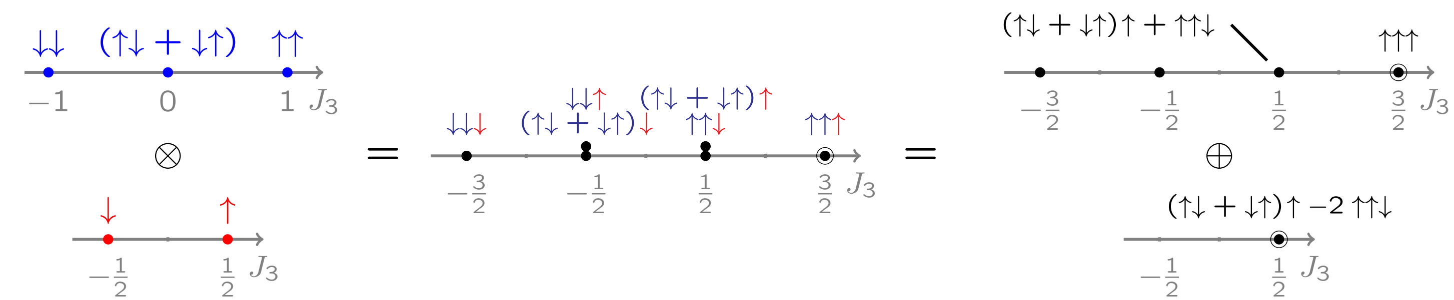
$J^-_{2\otimes 2}\,|\uparrow\uparrow\rangle = (J^-_2\,|\uparrow\rangle)\,|\uparrow\rangle + |\uparrow\rangle\,J^-_2\,|\uparrow\rangle \propto (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$



$$\boxed{\mathbf{2} \otimes \mathbf{2} = \mathbf{3}_s \oplus \mathbf{1}_a}$$

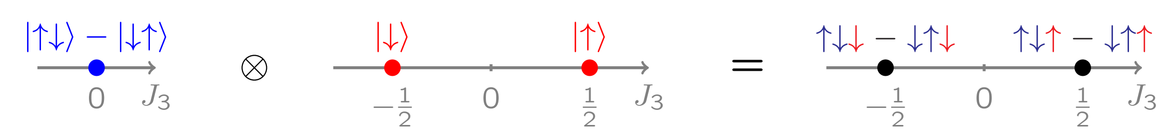
- in similar way we find

$$\mathbf{3}_s \otimes \mathbf{2} = \mathbf{4}_s \oplus \mathbf{2}_{m,s}$$



wave functions of the doublet have mixed symmetry (m,s) in the first two positions

- and trivially: $\mathbf{1}_a \otimes \mathbf{2} = \mathbf{2}_{m,a}$



wave functions of the doublet have mixed asymmetry (m,a) in the first two positions

- to summarize

$$\boxed{\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_s \oplus \mathbf{2}_{m,s} \oplus \mathbf{2}_{m,a}}$$

For reading and for exercises

- For reading (J. Chyla textbook)
  - chapters 2.1-2.4
- Exercises 3.1, 3.2, 3.17, 4.4