Generators, weights, roots and shifting operators of group SU(3). Adjoint representation.

**SU(3) generators**

Group SU(3) - 3×3 unitary (U) matrices with unit determinant (S - special). This is also a representation of the group on three dimensional Hilbert space. Generators are Hermitian matrices 3×3 with zero trace. Such general Hermitian matrix can be parametrized with eight real numbers $a, \ldots, h$:

$$
\begin{pmatrix}
    a & c - id & e - if \\
    c + id & b & g - ih \\
    e + if & g + ih & -a - b
\end{pmatrix}.
$$

In analogy with SU(2) and with Pauli matrices, we can take for the base of SU(3) Lie algebra the following generators, so called Gell-Mann matrices:

$$
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
$$

In general any 3×3 complex matrix can be expressed as (complex) linear combination of Gell-Mann matrices and unit matrix $\mathbf{1}$. Matrix 3×3 has nine complex numbers (in general for SU(n) it has $n^2$ complex parameters) and there are 8 generator (for SU(n) we have $n^2 - 1$) plus unit matrix, which gives also 9 terms ($n^2$), i.e. again 9 complex parameters. It is straightforward to find which linear combinations of Gell-Mann matrices are needed to produce desired matrix element. For the product of two Gell-Mann matrices we can therefor write:

$$
\lambda_a \cdot \lambda_b = A_{ab} \mathbf{1} + B_{abc} \lambda_c, \quad A_{ab}, B_{abc} \in \mathbb{C}.
$$

It is straightforward to check that the Gell-Man matrices give orthogonal basis of SU(3) algebra with normalization

$$
Tr \left( \lambda_a \cdot \lambda_b^\dagger \right) = 2\delta_{ab}.
$$
Canonical choice of normalization is then given for color matrices

\[ T_a \equiv \frac{\lambda_a}{2}, \quad \text{with} \quad \text{Tr} (T_a \cdot T_b^+) = \frac{1}{2} \delta_{ab}. \]

This gives

\[ T_a \cdot T_b = \frac{1}{6} \delta_{ab} \mathbf{1} + B_{abc} T_c, \quad B_{abc} \in \mathbb{C}. \]

It is easy to see that tensor $B$ stays constant under cyclic permutation of its parameters:

\[ \text{Tr} (T_a T_b T_c) = B_{abd} \text{Tr} (T_d T_e) = \frac{3}{2} B_{abe}, \quad \Rightarrow \quad B_{abc} = B_{bca} = B_{cab}. \]

Explicitly expressing this tensor by two real tensors $d$ and $f$:

\[ B_{abc} = \frac{1}{2} (d_{abc} + i f_{abc}) \quad d_{abc}, f_{abc} \in \mathbb{R}, \]

and evaluating

\[ (T_a T_b)^+ = \frac{1}{6} \delta_{ab} \mathbf{1} + \frac{1}{2} (d_{abc} - i f_{abc}) T_c^+ = \frac{1}{6} \delta_{ab} \mathbf{1} + \frac{1}{2} (d_{abc} - i f_{abc}) T_c, \]

which also equals to

\[ (T_a T_b)^+ = (T_b^+ T_a^+) = (T_b T_a) = \frac{1}{6} \delta_{ab} \mathbf{1} + \frac{1}{2} (d_{bac} + i f_{bac}) T_c, \]

we see immediately that $d_{abc}$ is totally symmetric while $f_{abc}$ is totally anti-symmetric. Evaluating the commutator between two color matrices leads to

\[ [T_a, T_b] = \frac{1}{2} (d_{abc} + i f_{abc}) T_c - \frac{1}{2} (d_{bac} + i f_{bac}) T_e = i f_{abc} T_c \]

i.e. $f_{abc}$ are indeed the structure constants. And for the anti-commutator we get

\[ \{T_a, T_b\} = \frac{1}{3} \delta_{ab} \mathbf{1} + \frac{1}{2} (d_{abc} + i f_{abc}) T_e + \frac{1}{2} (d_{bac} + i f_{bac}) T_c = \frac{1}{3} \delta_{ab} \mathbf{1} + d_{abc} T_c. \]

Finally we can write

\[ T_a \cdot T_b = \frac{1}{6} \delta_{ab} \mathbf{1} + \frac{1}{2} (d_{abc} + i f_{abc}) T_c. \]

This relation will be useful later on for the evaluation of traces of color matrices in calculations of QCD matrix elements without explicit knowledge of values of structure constants $f_{abc}$ and symmetric tensor $d_{abc}$. This derivation works also for general SU(n) fundamental representation.
Weights of fundamental SU(3) representation

The rank of group SU(3) is two as no more than two color matrices commute together. These are the two diagonalized generators $T_3$ and $T_8$, which form Cartan subalgebra with $H_1 = T_3$ and $H_2 = T_8$. The weights of the fundamental SU(3) representation are given by proper numbers of those two generators. As we have made them diagonal already, we can chose the base vectors as

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

We can find the proper numbers by applying $H_i$ to those vectors:

\[ T_3 e_1 = \frac{1}{2} e_1, \quad T_8 e_1 = \frac{1}{2\sqrt{3}} e_1 \quad \Rightarrow \quad \vec{\mu}_1 = \left( \frac{1}{2}, \frac{1}{2\sqrt{3}} \right). \]

These are basically diagonal members of $T_3$ and $T_8$. For $e_2$ and $e_3$, we get immediately

\[ \vec{\mu}_2 = \left( -\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \text{and} \quad \vec{\mu}_3 = \left( 0, -\frac{1}{\sqrt{3}} \right). \]

Figure 1: Base vectors of Hilbert space for SU(3) fundamental triplet.
SU(3) roots and shifting operators

Remaining six generators, outside of Cartan subalgebra, could be rearranged into six shifting operators $E_{\vec{\alpha}}$ satisfying relation

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}},$$

with $H_1 = T_3$ and $H_2 = T_8$.

We could follow the general prescription and find the roots $\vec{\alpha}$ and the shifting operators $E_{\vec{\alpha}}$ as proper numbers and proper vectors of Cartan subalgebra operators $H_i$. For this, we would need explicit knowledge of SU(3) structure constants. We can, however, guess the form of shifting operators, noting that pairs $T_1$ and $T_2$ has form of Pauli matrices.

Similarly, for $T_4$ and $T_5$, and $T_6$ and $T_7$. The shifting operators will have form

$$\frac{1}{\sqrt{2}} (T_1 \pm iT_2), \frac{1}{\sqrt{2}} (T_4 \pm iT_5), \frac{1}{\sqrt{2}} (T_6 \pm iT_7).$$

We can check these are indeed shifting operators and find the roots by explicit evaluation of the commutator with the members of Cartan subalgebra. For $(T_1 \pm iT_2)$ and $H_1 = T_3$ we know the results as this is exactly the same as in case of SU(2)

$$[T_3, (T_1 \pm iT_2)] = \pm 1 (T_1 \pm iT_2).$$

For $T_8$ we get

$$[T_8, (T_1 \pm iT_2)] = \frac{1}{4\sqrt{3}} \left[ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mp i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = 0.$$

We have found two roots

$$\begin{pmatrix} 1, 0 \end{pmatrix} (= \vec{\alpha}_3), \text{ and } \begin{pmatrix} -1, 0 \end{pmatrix} = (\vec{-\alpha}_3)$$

and corresponding shifting operators

$$E_{\pm \vec{\alpha}_3} = \frac{1}{\sqrt{2}} (T_1 \pm iT_2).$$

For $(T_4 + iT_5)$ we get

$$[T_3, (T_4 + iT_5)] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{4} \left[ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \frac{1}{2} (T_4 + iT_5),$$
\[ [T_8, (T_4 + iT_5)] = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ = \frac{1}{2\sqrt{3}} \left[ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \frac{\sqrt{3}}{2} (T_4 + iT_5). \]

We have found another root
\[ \vec{\alpha}_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \]
for shifting operator
\[ E_{\vec{\alpha}_1} = \frac{1}{\sqrt{2}} (T_4 + iT_5). \]

We will omit the vector notation for roots in subscript of the shifting operators. We note that the forth root is \(-\vec{\alpha}_1\) and corresponding shifting operator is \(E_{\vec{\alpha}_1}^+\) which is indeed \((T_4 - iT_5) / \sqrt{2}\).

Finally, let’s evaluate the commutators with Cartan subalgebra for \((T_6 - iT_7):\)

\[ [T_3, (T_6 - iT_7)] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \]
\[ = \frac{1}{4} \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \right] = \frac{1}{2} (T_6 - iT_7), \]

\[ [T_8, (T_6 - iT_7)] = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]
\[ = \frac{1}{2\sqrt{3}} \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] = -\frac{\sqrt{3}}{2} (T_6 - iT_7). \]

And we have remaining two roots
\[ \vec{\alpha}_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \text{ and } -\vec{\alpha}_2 = \left( \frac{-1}{2}, \frac{\sqrt{3}}{2} \right) \]

with shifting operators
\[ E_{\vec{\alpha}_2} = \frac{1}{\sqrt{2}} (T_6 - iT_5) \text{ and } E_{-\vec{\alpha}_2} = \frac{1}{\sqrt{2}} (T_6 + iT_5). \]
Figure 2: Roots of SU(3) group.

To summarize, SU(3) has six roots, three positive roots

\[ \vec{\alpha}_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad \vec{\alpha}_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad \vec{\alpha}_3 = (1, 0), \]

plus three negative directions \(-\vec{\alpha}_1, -\vec{\alpha}_2, -\vec{\alpha}_3\). Corresponding six shifting operators are then

\[ E_{\pm \alpha_1} = \frac{1}{\sqrt{2}} (T_4 \pm iT_5), \quad E_{\pm \alpha_2} = \frac{1}{\sqrt{2}} (T_6 \mp iT_7), \quad E_{\pm \alpha_3} = \frac{1}{\sqrt{2}} (T_1 \pm iT_2). \]

Knowing the weights of fundamental representation, we could have find out the values of six roots simply by taking the difference between all pairs of weights. For example \(\vec{\alpha}_3 = \vec{\mu}_1 - \vec{\mu}_2\), etc.

**Adjoint representation of SU(3)**

Adjoint representation lives on the Hilbert space with dimension equal to the number of group generators. In case of SU(3), the dimension of the Hilbert space of adjoint representation is therefore eight. We already now six states as the roots of Lie algebra are weights of states in adjoint representation. Remaining two base states correspond to the generators from the Cartan subalgebra. They do not shift the state, which would correspond to zero root and the weights of these two states are therefore zero.
This could be seen also by construction of Cartan subalgebra operators $H_i$, with $i = 1, 2, \ldots, r$, where $r$ is rank of the group. In adjoint representation, the matrix element is given in general as $(T_a)_{bc} \equiv -i f_{abc}$. If we order the generators in such a way that $H_i$ are the first on the list, then matrix $(H_i)_{bc}$ has zeros in first $r$ rows and columns, which follows from the antisymmetry of structure constants $f_{abc}$. All Cartan generators will therefore have $r$ degenerate states corresponding to zero proper number.