Excitonic Condensation of Strongly Correlated Electrons

Jan Kuneš

DFG FOR1346
Outline

- Excitonic condensation in fermion systems

- EC phase in the two-band Hubbard model (DMFT results)

- (Pr$_x$Ln$_{1-x})_y$Ca$_{1-y}$CoO$_3$ (PCCO)

- LDA+U for PCCO
  excitonic condensation with orbital degeneracy
A band insulator with a very narrow gap (positive or negative) is unstable towards opening of a gap due to electron-hole attraction - condensation of excitons.

The gap can have spin-singlet or spin-triplet symmetry and be real or imaginary. Which of these options is realised depends on the interaction term and details of the band structure.

*Mott, 1961*

*Halperin and Rice, 1968*
Two-orbital atom with 2 electrons (crystal field and Hund’s exchange determine the states of lowest energy):

- **Low spin** \( S=0 \)
- **High spin** \( S=1 \)

Strong coupling: HS states behave as hard-core bosons with the vacuum state \( |\text{vac}\rangle \equiv |\text{LS}\rangle \)

Bose-Einstein condensation = spontaneous hybridization between HS and LS states on the same site (breaks spin rotational symmetry)
Excitonic instability close to spin-state transition

Two-band Hubbard model at n=2 (half filling)

\[ H_\text{t} = \frac{\Delta}{2} \sum_{i,\sigma} \left( n_{i\sigma}^a - n_{i\sigma}^b \right) + \sum_{i,j,\sigma} \left( t_a a_{i\sigma}^\dagger a_{j\sigma} + t_b b_{i\sigma}^\dagger b_{j\sigma} \right) + \sum_{\langle ij \rangle,\sigma} \left( V_1 a_{i\sigma}^\dagger b_{j\sigma} + V_2 b_{i\sigma}^\dagger a_{j\sigma} + \text{c.c.} \right) \]

\[ H_\text{int}^{dd} = U \sum_i \left( n_{i\uparrow}^a n_{i\downarrow}^a + n_{i\downarrow}^b n_{i\uparrow}^b \right) + (U - 2J) \sum_{i,\sigma} n_{i\sigma}^a n_{i\sigma}^b 
+ (U - 3J) \sum_{i,\sigma} n_{i\sigma}^a n_{i\sigma}^b \]
\[ H_\text{int}' = J \sum_{i,\sigma} a_{i\sigma}^\dagger b_{i-\sigma} a_{i-\sigma} b_{i\sigma} + J' \sum_i \left( a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger b_{i\downarrow} b_{i\uparrow} + \text{c.c.} \right) . \]

- Full p-h susceptibility tensor \( \chi_q(T) \) for various temperatures
- Investigate dominant eigenmodes of \( \chi_q(T) \)

**orbital diagonal** \( \sum_{\sigma} n_{\sigma}^a - n_{\sigma}^b \)

**orb. off-diagonal** \( a_{\uparrow}^\dagger b_{\downarrow}, a_{\downarrow}^\dagger b_{\uparrow}, b_{\uparrow}^\dagger a_{\downarrow}, b_{\downarrow}^\dagger a_{\uparrow} \)

**spin longitudinal** \( \sum_{\sigma} \sigma \left( n_{\sigma}^a + n_{\sigma}^b \right) \)
**Excitonic instability close to spin-state transition**

Two-band Hubbard model at $n=2$ (half filling)

\[
H_t = \frac{\Delta}{2} \sum_{i, \sigma}(n_{i \sigma}^a - n_{i \sigma}^b) + \sum_{i, j, \sigma}(t_a a_{i \sigma}^\dagger a_{j \sigma} + t_b b_{i \sigma}^\dagger b_{j \sigma}) \\
+ \sum_{(ij), \sigma}(V_1 a_{i \sigma}^\dagger b_{j \sigma} + V_2 b_{i \sigma}^\dagger a_{j \sigma} + c.c.)
\]

\[
H_{int}^{dd} = U \sum_i (n_{i \uparrow}^a n_{i \downarrow}^a + n_{i \downarrow}^b n_{i \uparrow}^b) + (U - 2J) \sum_{i, \sigma} n_{i \sigma}^a n_{i \sigma}^b \\
+ (U - 3J) \sum_{\sigma} n_{i \sigma}^a n_{i \sigma}^b
\]

\[
H_{int}' = J \sum_{i, \sigma} a_{i \sigma}^\dagger b_{i \sigma}^\dagger a_{i \sigma} b_{i \sigma} + J' \sum_{i} (a_{i \uparrow}^\dagger a_{i \downarrow}^\dagger b_{i \uparrow} b_{i \downarrow} + c.c.).
\]

- Full p-h susceptibility tensor $\chi_q(T)$ for various $T$
- Investigate dominant eigenmodes of $\chi_q(T)$
Excitonic instability close to spin-state transition

Two-band Hubbard model at $n=2$ (half filling)

$$H_t = \frac{\Delta}{2} \sum_{i,\sigma}(n_{i\sigma}^a - n_{i\sigma}^b) + \sum_{i,j,\sigma}(t_a a_{i\sigma}^\dagger a_{j\sigma} + t_b b_{i\sigma}^\dagger b_{j\sigma})$$

$$+ \sum_{(ij),\sigma}(V_1 a_{i\sigma}^\dagger b_{j\sigma} + V_2 b_{i\sigma}^\dagger a_{j\sigma} + c.c.)$$

$$H_{\text{int}}^{\text{dd}} = U \sum_i (n_{i\uparrow}^a n_{i\downarrow}^a + n_{i\downarrow}^b n_{i\uparrow}^b) + (U - 2J) \sum_{i,\sigma} n_{i\sigma}^a n_{i\sigma}^b$$

$$+ (U - 3J) \sum_{i,\sigma} n_{i\sigma}^a n_{i\sigma}^b$$

$$H_{\text{int}}' = J \sum_{i,\sigma} a_{i\sigma}^\dagger b_{i\sigma}^\dagger a_{i\sigma} - b_{i\sigma} + J' \sum_i (a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger b_{i\downarrow} + c.c.).$$

- Full p-h susceptibility tensor $\chi_q(T)$ for various temperatures
- Investigate dominant eigenmodes of $\chi_q(T)$

**orbital diagonal** $\sum_{\sigma} n_{\sigma}^a - n_{\sigma}^b$

**orb. off-diagonal** $a_{\downarrow}^\dagger b_{\downarrow}^\dagger, a_{\uparrow}^\dagger b_{\uparrow}^\dagger, b_{\downarrow}^\dagger a_{\downarrow}, b_{\uparrow}^\dagger a_{\uparrow}$

**spin longitudinal** $\sum_{\sigma} (n_{\sigma}^a + n_{\sigma}^b)$
Excitonic instability

$U=4$, $J=1$

$t_{a}^{2}+t_{b}^{2} = \text{const}$

$\Delta = 3.40$

$V=0$

$\zeta = \frac{2t_{a}t_{b}}{t_{a}^{2}+t_{b}^{2}}$

$\chi(q) \text{ for } \sigma_{a} \parallel \sigma_{b}$

$\Delta$ from Ref. 11. The blue line marks the estimated position of

square marks the position of the reentrant transition taken

Temperature (K)

normal phase

excitonic insulator (superfluid)

solid order

$\lambda_{i}$,...

$J_{K}$ and Augustinsky, 2014
Excitonic instability

$U=4$, $J=1$
$t_a^2 + t_b^2 = \text{const}$
$\Delta = 3.40$
$V=0$

$\zeta = \frac{2t_a t_b}{t_a^2 + t_b^2}$.
Excitonic instability

\[ U=4, \; J=1 \]
\[ t_a^2 + t_b^2 = \text{const} \]
\[ \Delta = 3.40 \]
\[ V=0 \]
\[ \zeta = \frac{2t_at_b}{t_a^2 + t_b^2}. \]

\[ \text{Divergent response to:} \quad a_\sigma^\dagger b_{-\sigma} \]

\[ \text{excitonic insulator (superfluid)} \]

\[ \text{normal phase} \]

\[ \text{solid order} \]

JK and Augustinsky, 2014
Excitonic condensate - oder parameter

Order parameter is a complex vector: \[ \phi^\gamma = \sum_{\alpha\beta} \sigma_{\alpha\beta} \langle a_{\alpha}^\dagger b_{\beta} \rangle \]

z-axis parallel to: \[ i(\bar{\phi} \wedge \phi) \]

No cross hopping (V=0) -> overall phase of \( \phi \) does not matter
- phases are distinguished by \( |\phi^+| \) and \( |\phi^-| \)

3 possible phases:

linear (L) \[ |\phi^+| = |\phi^-| \neq 0 \]

elliptic (E) \[ 0 \neq |\phi^+| \neq |\phi^-| \neq 0 \]

circular (C) \[ |\phi^+| = 0, |\phi^-| \neq 0 \]
Excitonic condensate - oder parameter

Order parameter is a complex vector: \( \phi^\gamma = \sum_{\alpha\beta} \sigma_{\alpha\beta}^\gamma \langle a_\alpha^\dagger b_\beta \rangle \)

z-axis parallel to: \( i(\bar{\phi} \wedge \phi) \)

No cross hopping (V=0) -> overall phase
- phases are dis

3 possible phases:

linear (L) \( |\phi^+| = |\phi^-| \neq 0 \)

elliptic (E) \( 0 \neq |\phi^+| \neq |\phi^-| \neq 0 \)

circular (C) \( |\phi^+| = 0, |\phi^-| \neq 0 \)
Order parameter is a complex vector: \( \phi^\gamma = \sum_{\alpha\beta} \sigma^\gamma_{\alpha\beta} \langle a^\dagger_{\alpha} b_{\beta} \rangle \)

z-axis parallel to: \( i(\bar{\phi} \wedge \phi) \)

No cross hopping (V=0) -> overall phase of \( \phi \) does not matter

-> phases are distinguished by \( |\phi^+| \) and \( |\phi^-| \)

3 possible phases:

linear (L) \( |\phi^+| = |\phi^-| \neq 0 \)

elliptic (E) \( 0 \neq |\phi^+| \neq |\phi^-| \neq 0 \)

circular (C) \( |\phi^+| = 0, |\phi^-| \neq 0 \)
Excitonic instability

\[ U=4, \, J=1 \]
\[ t_a^2 + t_b^2 = \text{const} \]
\[ \Delta = 3.40 \]
\[ V=0 \]
\[ \zeta = \frac{2t_a t_b}{t_a^2 + t_b^2} \]

\[ \text{Temperature (K)} \]

\[ \text{normal phase} \]
\[ \text{excitonic insulator (superfluid)} \]

\[ \text{solid order} \]

\[ \text{JK and Augustinsky, 2014} \]
Excitonic condensation (n=2)

order parameter $\phi = \langle a_\uparrow^{\dagger}b_\downarrow + a_\downarrow^{\dagger}b_\uparrow \rangle$
Excitonic condensation (n=2)

order parameter \( \phi = \langle a_\uparrow^\dagger b_\downarrow + a_\downarrow^\dagger b_\uparrow \rangle \)

Optical conductivity (dc resistivity)
Excitonic condensation (n=2)

order parameter $\phi = \langle a^\dagger b_\downarrow + a^\dagger b_\uparrow \rangle$

Dynamical spin susceptibility (local)
Excitonic condensation (n=2)

order parameter \( \phi = \langle a_\uparrow b_\downarrow + a_\downarrow b_\uparrow \rangle \)

Dynamical spin susceptibility (local)

Local picture

Normal phase

EC phase

HS-LS hybridization
Excitonic condensation (doping) - all phases

n-T phase diagram

\[ T(K) \]

\[ n_h \]

L phase \[ |\phi^+| = |\phi^-| \]

E phase \[ |\phi^+| \neq |\phi^-| \]

C phase \[ |\phi^+| = 0 \]

\[ n_h - \text{hole concentration (N=2-n_h)} \]
Pr$_{0.5}$Ca$_{0.5}$CoO$_3$ (PCCO)

specific heat:

resistivity:

inverse susceptibility:

1. continuous phase transition
2. metal-insulator transition
3. drop of magnetic susceptibility

Tsubouchi et al., 2002
Hejtmanek et al., 2013
Pr\(_{0.5}\)Ca\(_{0.5}\)CoO\(_{3}\) (PCCO)

4. Pr\(^{3+}\)->Pr\(^{4+}\) valence transition
5. exchange splitting of Pr\(^{4+}\) ground state, but no magnetic order detected

Hejtmanek et al., 2013
Excitonic condensation (doping) - all phases

n-T phase diagram

- **L phase** \(|\phi^+| = |\phi^-|\)
- **E phase** \(|\phi^+| \neq |\phi^-|\)
- **C phase** \(|\phi^+| = 0\)

**FIG. 2:**
- (a) The order parameter \(\phi = \langle a^{\dagger} \uparrow b \downarrow + a^{\dagger} \downarrow b \uparrow \rangle\) as a function of temperature for stoichiometric filling \(n = 2\) (black) and at fixed chemical potential corresponding to hole doping between 0.03 and 0.12 (red to violet) in the normal phase.
- (b) Number of electrons per atom at fixed chemical potential across \(T_c\) (the same color coding as in (a)).
- (c) Top, the magnetic susceptibility \(\chi\) as a function of temperature at the stoichiometric filling (circles with error bars). The dotted line shows \(\chi(T)\) in the normal phase. The stars correspond to solutions below \(T_c\) constrained to the normal phase. The shaded area marks the EC phase.
- Bottom, the same as above in the system with fixed chemical potential and hole doping of 0.12 in the normal phase.

1. which leads to opening of a gap in the one-particle spectral density, as shown in Fig. 3. Opening of a gap naturally affects the optical conductivity shown in Fig. 3. Below \(T_c\), the Drude peak is rapidly destroyed, as the spectral weight is pushed to higher frequencies, and the resistivity grows exponentially. The spin susceptibility \(\chi_S\), Fig. 2, changes from the Curie-Weiss behavior above \(T_c\), reflecting the presence of thermal excited states, to a \(T\)-independent van Vleck paramagnetism arising from the on-site hybridization between LS and HS states represented by the off-diagonal self-energy. This effect is quite different from the spin-state transition characterized by vanishing of the HS population, which can be detected by x-ray absorption. In the E phase, the HS state remains populated and so the x-ray signature of the spin-state transition is missing. The sign of the \(\chi_S\) jump at \(T_c\) depends on details of the system. In Fig. 2 we demonstrate that when \(T_c\) is reduced by doping \(\chi_S\) is reduced below \(T_c\). The results show that excitonic condensation...
EC in cubic d⁶ perovskite

Exciton = bound pair of e_g electron and t_2g hole

How do we detect the EC order?

Local d-occupation matrix (10 x 10):

spin structure: \( D = \begin{pmatrix} D_0 + \phi^z & \phi^x + i\phi^y \\ (\phi^x + i\phi^y)^* & D_0 - \phi^z \end{pmatrix} \)

orbital structure:

\[ \phi_{xy} \alpha d_{x^2-y^2} \otimes d_{xy} + \phi_{zx} \alpha d_{z^2-x^2} \otimes d_{xz} + \phi_{yz} \alpha d_{y^2-z^2} \otimes d_{yz} \]

explicit form in spherical harmonic basis (real \( \Phi \)):

\[ \begin{pmatrix} 0 & -\frac{1}{4} (\phi'_{zx} + i\phi'_{yz}) & 0 & \frac{1}{4} (\phi'_{zx} - i\phi'_{yz}) & i\phi'_{xy} \\ -\frac{1}{4} (\phi'_{zx} - i\phi'_{yz}) & 0 & \sqrt{\frac{3}{8}} (\phi'_{zx} + i\phi'_{yz}) & 0 & -\frac{1}{4} (\phi'_{zx} - i\phi'_{yz}) \\ 0 & \sqrt{\frac{3}{8}} (\phi'_{zx} - i\phi'_{yz}) & 0 & -\sqrt{\frac{3}{8}} (\phi'_{zx} + i\phi'_{yz}) & 0 \\ \frac{1}{4} (\phi'_{zx} + i\phi'_{yz}) & 0 & -\sqrt{\frac{3}{8}} (\phi'_{zx} - i\phi'_{yz}) & 0 & \frac{1}{4} (\phi'_{zx} + i\phi'_{yz}) \\ -i\phi'_{xy} & -\frac{1}{4} (\phi'_{zx} + i\phi'_{yz}) & 0 & \frac{1}{4} (\phi'_{zx} - i\phi'_{yz}) & 0 \end{pmatrix} \]
EC in cubic d⁶ perovskite

Exciton = bound pair of e_g electron and t_2g hole

How do we detect the EC order?

Local d-occupation matrix (10 x 10):

spin structure: \[ \mathbf{D} = \begin{pmatrix} D_0 + \phi_z & \phi^z + i\phi^y \\ (\phi^x + i\phi^y)^* & D_0 - \phi^z \end{pmatrix} \]

orbital structure:

\[ \phi_{\alpha \beta}^{\text{orb}} = \phi_{x y}^{\alpha} d_{x^2-y^2} \otimes d_{xy} + \phi_{z x}^{\alpha} d_{z^2-x^2} \otimes d_{zx} + \phi_{y z}^{\alpha} d_{y^2-z^2} \otimes d_{yz} \]

The order parameter has 9 components (or 18 real components)

\[ \phi_{\alpha \beta} \]
\[ \alpha = x, y, z \text{ transforms like a vector under spin rotations} \]
\[ \beta = x, \hat{y}, \hat{z} \text{ transforms like a pseudovector under O_h operations} \]

The spin and orbital symmetry does not specify the ordered phase uniquely, possible solutions can be classified by their residual symmetry.
Examples of LDA+U EC solutions

LaCoO$_3$ AF-EC order

(a) (b) (c)

$\phi^\alpha_\beta$

$\begin{pmatrix}
X & 0 & 0 \\
0 & X & 0 \\
0 & 0 & X
\end{pmatrix}$

$\begin{pmatrix}
0 & 0 & X \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$

$\begin{pmatrix}
0 & 0 & X \\
0 & 0 & X \\
0 & 0 & X
\end{pmatrix}$

$\begin{pmatrix}
X & 0 & 0 \\
0 & X & 0 \\
0 & 0 & X
\end{pmatrix}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$X$</th>
<th>$E_{(i)}$ [meV/f.u.]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.182</td>
<td>-43</td>
</tr>
<tr>
<td>2</td>
<td>0.134</td>
<td>-73</td>
</tr>
<tr>
<td>3</td>
<td>0.144</td>
<td>-82</td>
</tr>
</tbody>
</table>

TABLE III: The orbital parts of the EC order parameter for the fershared spheres (shown in Table III). Given their small size we do not calculate the orbital space. We have verified that these solutions are stable and have the same amplitudes of the order parameter, total energies and total residual symmetry.

\[ \text{TABLE III: } \begin{pmatrix}
X & 0 & 0 \\
0 & X & 0 \\
0 & 0 & X
\end{pmatrix} \]

Our goal is not to investigate the one-particle densities of states. As done in LSDA+U, the one-particle densities of states changes when in the spin rotational symmetry. The residual symmetry group contains the elements of the spin rotational symmetry. Finally, we have found a non-product solution, which has only $\phi$ and $\beta$.

Calculations must be performed in LDA+U and not LSDA+U mode.
Examples of LDA+U for PCCO

orthorombic structure: 4 Co atoms per f.u. two inequivalent Co positions

Product solution:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{yz}$</td>
<td>0.182</td>
<td>0.182</td>
<td>0.216</td>
<td>0.216</td>
</tr>
<tr>
<td>$\phi_{zx}$</td>
<td>0.228</td>
<td>0.228</td>
<td>-0.212</td>
<td>-0.212</td>
</tr>
<tr>
<td>$\phi_{xy}$</td>
<td>-0.071</td>
<td>0.071</td>
<td>-0.093</td>
<td>0.093</td>
</tr>
</tbody>
</table>

Orbital pseudovectors on sym. related Co atoms:
Origin of exchange splitting on Pr

Coupling of Pr 4f^1 spin to p-d orbitals: effective multi-channel Kondo Hamiltonian

\[ H^{(n)} = \sum_{\alpha \alpha'} \sum_{mm'} \sum_i S \cdot \sigma_{\alpha \alpha'} J_{i,mm'}^{(n)} c_{im\alpha}^\dagger c_{im'\alpha'} + c.c. \]

Below T_c effective exchange field appears: \[ h^{(n)}_{\gamma} = \sum_{imm'} J_{i,mm'}^{(n)} \sum_{\alpha \alpha'} 2 \text{Re} \langle c_{im\alpha}^\dagger \sigma_{\alpha \alpha'}^\gamma c_{im'\alpha'} \rangle \]

The site symmetry of the EC order parameter with respect to the Pr site decides whether contributions of from different Co site interfere constructively or destructively.

For the present EC solution h=0 in the absence of spin-orbit coupling in Pr 4f shell. With SOC splitting on 10 meV scale is obtained.
Conclusions

• Solids close to spin-state transition may be unstable towards condensation of spinful excitons.

• The excitonic condensate breaks the spin rotation symmetry.

• The excitonic condensation can lead to a long-range order of magnetic multipoles (but there are other possibilities as well).

Solid order - strongly asymmetric bands

HS are immobile (classical BEG model). The physics is dominated by HS-HS repulsion and HS-HS anti-ferromagnetic exchange.

\[
\tilde{H} = D \sum_i s_i^2 + K \sum_{\langle ij \rangle} s_i^2 s_j^2 + I \sum_{\langle ij \rangle} s_i s_j
\]

JK and Krapek, 2011
The order parameter \( \phi = \langle a_{\uparrow}^\dagger b_{\downarrow} + a_{\downarrow}^\dagger b_{\uparrow} \rangle \)

Spectral density (diagonal elements)

Self-energy

Green’s function

Excitonic condensation (n=2)
Excitonic condensation (n=2)

order parameter $\phi = \langle a_\uparrow b_\downarrow + a_\downarrow b_\uparrow \rangle$

Spectral density (diagonal elements)
Excitonic condensation (doping) - all phases

n-T phase diagram

μ-T phase diagram

L phase  \(|\phi^+| = |\phi^-|\)
E phase  \(|\phi^+| \neq |\phi^-|\)
C phase  \(|\phi^+| = 0\)

\(n_h\) - hole concentration (N=2-\(n_h\))
Excitonic condensation (fixed $\mu$)

\[ \phi = \langle a_{\uparrow}^\dagger b_{\downarrow} + a_{\downarrow}^\dagger b_{\uparrow} \rangle \]

\[ n(T) = \begin{cases} \text{const} & T > T_c \\ \rightarrow 2 & T < T_c \end{cases} \]
Excitonic condensation (fixed $\mu$)

$$\phi = \langle a_{\uparrow}^\dagger b_{\downarrow} + a_{\downarrow}^\dagger b_{\uparrow} \rangle$$

Spin susceptibility

The order parameter $\phi$ grows exponentially. The spin susceptibility rapidly destroys as the spectral weight is pushed to high, signifying the vanishing of the HS population, which can be detected by x-ray absorption. In the normal phase, paramagnetism arising from the on-site hybridization is quite different. The results so far show that excitonic condensation ($\phi$) increases from the Curie-Weiss behavior ($\chi$) as a function of temperature at the stoichiometric filling (circles with error bars). The dotted line shows the expected weight.

$\chi_S$ (electrons per atom)

$\chi_C$ (electrons per atom)

$\chi_S$ corresponds to solutions below $T_c=2$ (black) and at fixed chemical potential corresponding to hole doping ($0.2$ to $1.95$). The Drude peak is quite different from the electronic and hole doping of $0.03$ to $0.12$ (red to violet) in the normal phase. The stars in the electronic phase ($\chi_S$) are shown in Fig. 2, changes from the Curie-Weiss behavior ($\chi_C$). The results so far show that excitonic condensation ($\phi$) increases from the Curie-Weiss behavior ($\chi$) as a function of temperature at the stoichiometric filling (circles with error bars). The dotted line shows the expected weight.

$n=2$

Susceptibility ($\mu_B^2$/eV)

fixed $\mu$

<table>
<thead>
<tr>
<th>$n(T)$</th>
<th>$T&gt;T_c$</th>
<th>$T&lt;T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>const</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rightarrow 2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
LaCoO$_3$ AF-EC orbitals (c) coupled to the orbital part as proper rotations, coupled to the corresponding spin rotation and to the orbital part unaltered, and the elements of $O_h$, which admits ferro and G-type antiferro-orbitals. We have verified that these solutions with the residual symmetry factorize, which change the orbital space. We have saved two such solutions. The first solution is a non-product solution, which has only the one-particle densities of states. As the existence of some, the calculations linear Wien2k code breaks by construction. However, this type of calculation with the spin-collinear approximation always converged to one of these solutions. We want to provide examples of several stable solutions. The second solution we have found for the order parameter factorizes, with the same amplitudes of the order parameter, total energies and these solutions. The first solution we have found for the order parameter factorizes, with the same amplitudes of the order parameter, total energies and these solutions. The first solution we have found for the order parameter factorizes, with the same amplitudes of the order parameter, total energies and these solutions. The first solution we have found for the order parameter factorizes, with the same amplitudes of the order parameter, total energies and these solutions.