Lectures on Optics

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Contents

1. Introduction

1.1 Electromagnetic spectrum

The electromagnetic radiation propagating in space is called *monochromatic* if it is characterized by a single frequency ν, a single angular frequency ω ($\omega = 2\pi\nu$) or a single wavelength λ ($\lambda = c/\nu$). If the electromagnetic field transmits several frequencies (wavelengths) either discrete or continuously distributed it is called *polychromatic*. The electromagnetic energy is distributed among these components; this distribution is called the spectrum of the radiation. The whole accessible electromagnetic spectrum is divided into a number of spectral regions which have particular names, like radio-waves, micro-waves, optical waves, etc. We distinguish between these regions because they are usually suitable for different applications; the emission and the detection of the electromagnetic field in these different spectral ranges are often based on different physical processes. A schematic view of the electromagnetic spectrum, which is exploited in industrial or scientific applications, is given in Fig. 1.1. The radiation within the narrow range between 380 nm and 770 nm is able to produce visual sensations in the human eye and it is called "light". This range is limited at both sides by an invisible radiation: ultraviolet (short wavelength side) and infrared (long wavelength side). The three spectral bands make together the optical range which will be of particular interest in this textbook.

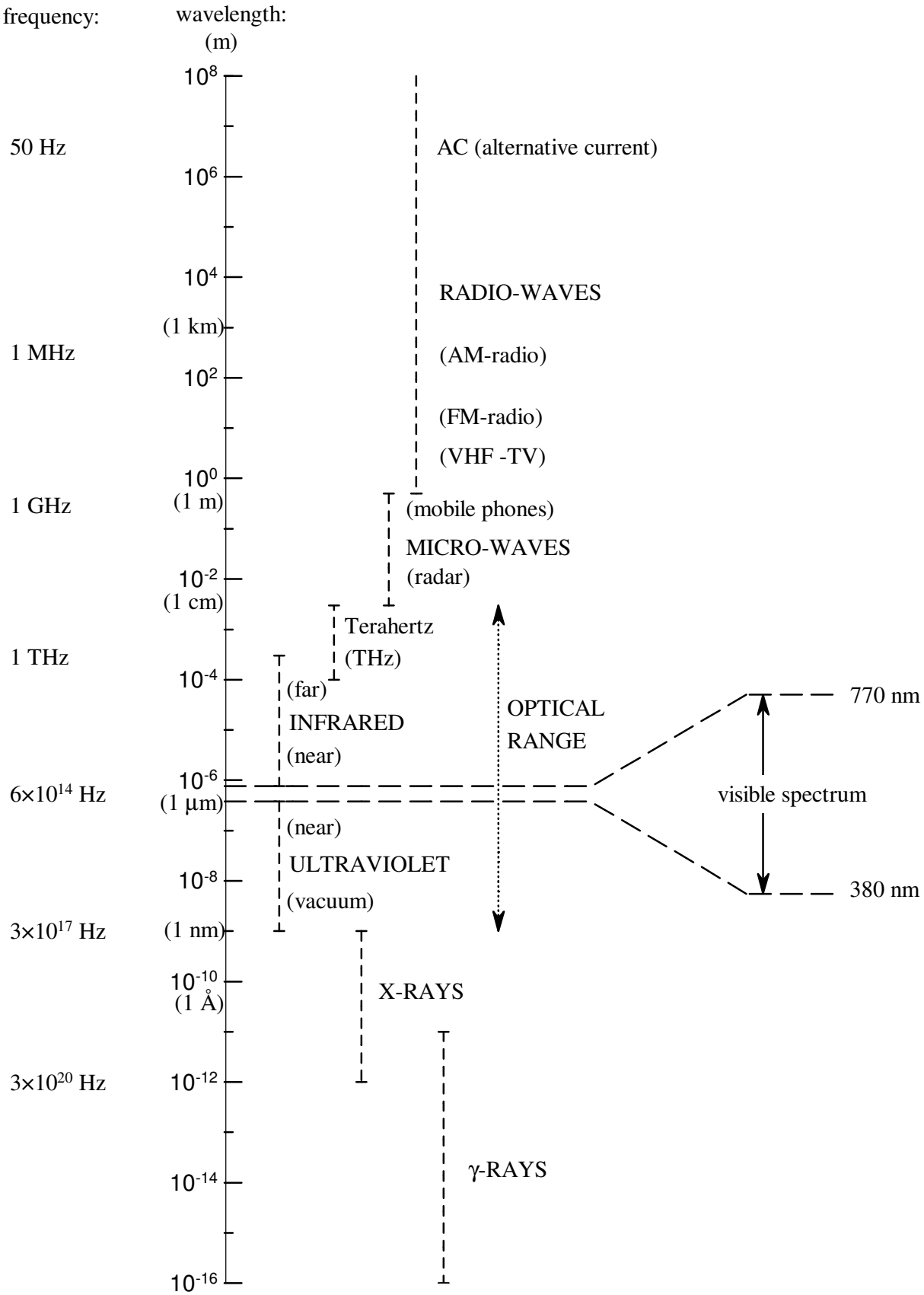

Figure 1.1: Overview of the electromagnetic spectrum.

A. Complex formalism, spectral decomposition of the fields

In optics we meet both the monochromatic and the polychromatic radiation. As it was already pointed out, the monochromatic wave contains only one spectral component. Then the fields exhibit a harmonic variation in time:

$$a(t) = |A| \cos(\omega t + \alpha) \tag{1.1}$$

where ω is the angular frequency of the wave and α is its initial phase. The function $a(t)$ represents symbolically one of the electromagnetic field vectors (electric field or magnetic field) and $|A|$ is the amplitude of the harmonic oscillation. It is convenient to represent this wave in a complex notation. Eq. (1.1) then reads:

$$a(t) = \mathrm{Re}\{A e^{i\omega t}\} = \frac{1}{2}\left(A\, e^{i\omega t} + A^*\, e^{-i\omega t}\right), \text{ where } A = |A| e^{i\alpha}. \tag{1.2}$$

In order to simplify the calculations we will often write simply:

$$\overline{a}(t) = A\, e^{i\omega t} \tag{1.3}$$

instead of (1.1). $\overline{a}(t)$ is not exactly equal to $a(t)$ because the expression for $\overline{a}(t)$ has a non-vanishing imaginary part, but we will keep in mind when writing such an expression that only its real part has a physical sense. The complex representation does not bring any difficulty as far as only linear mathematical operations are involved in the calculations (*e.g.* linear combinations, Fourier transformation, *etc.*): the whole calculation can be performed in the complex representation [*i.e.* for the quantity $\overline{a}(t)$]; the real part of the result will then correspond to the calculated real physical quantity. Sometimes, physical quantities depending on higher powers of the fields are to be calculated (*e.g.* the electromagnetic energy): these cases require a careful treatment and will be always pointed out in this textbook.

A polychromatic wave can be described as a linear combination of spectral components. It can contain several discrete components of the type (1.1):

$$a(t) = \sum_k |A_k| \cos(\omega_k t + \alpha_k) \tag{1.4}$$

or, in the general case, it can contain a continuous distribution of spectral components characterized by their amplitudes $|A(\omega)|$ and by their phases $\alpha(\omega)$:

$$a(t) = \int_0^\infty |A(\omega)| \cos(\omega t + \alpha(\omega))\, d\omega. \tag{1.5}$$

We can then use the complex formalism in the same manner as for (1.3):

$$\overline{a}(t) = \int_0^\infty A(\omega) e^{i\omega t}\, d\omega\,, \text{ where } A(\omega) = |A(\omega)| e^{i\alpha(\omega)}. \tag{1.6}$$

We recall that only the real part of $\overline{a}(t)$ has a physical sense. The equation (1.6) is similar to the Fourier transformation definition (see Appendix A). One should define conveniently $A(\omega)$ for negative frequencies in order to obtain the Fourier-transformation relation between $\overline{a}(t)$ and $A(\omega)$. There are two simple possibilities how to proceed:

(1) Let us define $A(\omega < 0) = 0$. Then

$$\overline{a}(t) = \int_{-\infty}^{\infty} A(\omega) e^{i\omega t}\, d\omega \tag{1.7}$$

is complex, and the real quantity $a(t)$ is obtained as:

$$a(t) = \mathrm{Re}\{ \int_{-\infty}^{\infty} A(\omega) e^{i\omega t}\, d\omega \}\,. \tag{1.8}$$

(2) Let us define $A(-\omega) = A^*(\omega)$. Then

$$a(t) = \frac{1}{2} \int_{-\infty}^{\infty} A(\omega) e^{i\omega t}\, d\omega\,. \tag{1.9}$$

The result is then automatically real and corresponds directly to the calculated physical quantity.

We will sometimes need to calculate the mean value of a product of two harmonic functions (as we will see later the density of the electromagnetic energy and its flow are given by such a product). We obtain in this case:

$$\langle a(t)b(t)\rangle = \frac{1}{T} \int_0^T |A| \cos(\omega t + \alpha)\, |B| \cos(\omega t + \beta)\, dt = \frac{1}{2} |AB|\, \cos(\alpha - \beta) \tag{1.10}$$

where $T = 2\pi/\omega$ is the period of the harmonic signal. The terms of the signal oscillating with ω and 2ω were removed owing to the temporal average. If we introduce the complex amplitudes A and B in a similar manner as we did for (1.2), we easily obtain for the mean value of the product:

$$\langle a(t)b(t)\rangle = \frac{1}{2} \mathrm{Re}\{AB^*\}\,. \tag{1.11}$$

The rapidly oscillating terms cancel out due to the complex conjugation of the quantity B. The complex formalism can thus be safely applied to the evaluation of the mean values of bilinear forms.

1.2 Basis of the electromagnetic field theory

A. Maxwell equations

Differential form The propagation of light is described by the Maxwell equations (ME): the electromagnetic field is described by the electric ($\boldsymbol{E}$) and the magnetic ($\boldsymbol{H}$) field vectors which may vary in space and in time. The interaction with the matter can be conveniently described if a second pair of vectors — $\boldsymbol{D}$ (electric induction) and $\boldsymbol{B}$ (magnetic induction) — is introduced. These four vectors are connected through the Maxwell equations (MKS system of units):

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{1.12a}$$

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j} \tag{1.12b}$$

$$\nabla \cdot \boldsymbol{D} = \rho \tag{1.12c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.12d}$$

where $\boldsymbol{j}$ (density of electric current) and ρ (density of free electric charges) represent the sources of the electromagnetic field.

Integral form The four ME represent the fundamental laws of electricity and magnetism. Their physical sense becomes more clear if they are written in their integral form:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \int \boldsymbol{B} \cdot d\boldsymbol{S} \tag{1.13a}$$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \frac{\partial}{\partial t} \int \boldsymbol{D} \cdot d\boldsymbol{S} + \boldsymbol{J} \tag{1.13b}$$

$$\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q \tag{1.13c}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{S} = 0 \tag{1.13d}$$

Equation (1.13a) describes the Faraday's law of the electromagnetic induction: a change of the magnetic flux induces a vortex electric field; equation (1.13b) expresses the generalized Ampère's law which describes the induction of the magnetic field by a motion of charges: free charges contribute to the current $\boldsymbol{J}$; bound charges (called also polarization charges) form the Maxwell's displacement current. Eq. (1.13c) is equivalent to the Coulomb's law and recognizes the electric charges as sources of the electric field, and finally, (1.13d) postulates the absence of magnetic charges.

Boundary conditions In optics we often deal with the situation where the light propagates through an interface (discontinuity) between two different media. In this case it is useful to write the ME in the form of boundary conditions describing the continuity or the discontinuity of normal (index n) or tangential (index t) components of the field vectors at the interface:

$$\boldsymbol{E}_{2t} - \boldsymbol{E}_{1t} = 0 \tag{1.14a}$$
$$\boldsymbol{H}_{2t} - \boldsymbol{H}_{1t} = \boldsymbol{j}_s \tag{1.14b}$$
$$D_{2n} - D_{1n} = \sigma \tag{1.14c}$$
$$B_{2n} - B_{1n} = 0\,, \tag{1.14d}$$

where $\boldsymbol{j}_s$ is a surface current density, and σ is a surface charge density at the interface. However, it is clear that these surface quantities reflect an idealized concept. For example, if we place a good conductor into a static electric field, the free charges in the conductor will move to the positions on the surface in which they compensate the field inside the conductor: as a result the field in the bulk of the sample vanishes. The same is true for an ideal conductor (with an infinite conductivity) in an external magnetic field: electric currents are induced on the sample surface and these currents then screen the magnetic field which cannot thus penetrate into the bulk. The concept of surface currents and charge densities allows one to account for these ideal cases very simply: the quantities D_{2n} and $\boldsymbol{H}_{2t}$ from eqs. (1.14) vanish inside the sample, which means that the values of D_{1n} and $\boldsymbol{H}_{1t}$ are just compensated by the surface current and charge density.

On the other hand, in frame of these lectures we will treat propagative high-frequency (*i.e.* optical) electromagnetic fields: in this case a non-vanishing penetration depth of the electromagnetic field for any material exists and the induced charges and currents always have a bulk character ($\boldsymbol{j}_s = 0$, $\sigma = 0$). We will thus use the following interface conditions:

$$\boldsymbol{E}_{2t} - \boldsymbol{E}_{1t} = 0\,, \tag{1.15a}$$
$$\boldsymbol{H}_{2t} - \boldsymbol{H}_{1t} = 0\,, \tag{1.15b}$$
$$D_{2n} - D_{1n} = 0\,, \tag{1.15c}$$
$$B_{2n} - B_{1n} = 0\,; \tag{1.15d}$$

they express the continuity of the tangential components ($\boldsymbol{E}$ and $\boldsymbol{H}$) and of the normal components ($\boldsymbol{D}$ and $\boldsymbol{B}$) of the field vectors at the interface.

Constitutive relations The ME are supplemented by constitutive relations which describe the electromagnetic properties of the matter by means of the dielectric permittivity ϵ and magnetic permeability μ:

$$\boldsymbol{D} = \epsilon\boldsymbol{E} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{1.16}$$
$$\boldsymbol{B} = \mu\boldsymbol{H} = \mu_0\boldsymbol{H} + \boldsymbol{M}\,, \tag{1.17}$$

where $\boldsymbol{P}$ is the polarization vector and $\boldsymbol{M}$ is the magnetization vector; ϵ_0 and μ_0 are the vacuum permittivity and vacuum permeability, respectively. In optics, we are interested usually in the materials which are not magnetic, *i.e.* in the cases of $\mu = \mu_0$. The dielectric permittivity ϵ remains then the only quantity which characterizes the optical properties of materials. We can then classify the materials (and phenomena) following the properties of ϵ:

- anisotropic (ϵ is a second-rank tensor) — isotropic (ϵ is a scalar)

- inhomogeneous [$\epsilon = \epsilon(\boldsymbol{r})$] — homogeneous ($\epsilon$ is independent of $\boldsymbol{r}$)
- absorbing or opaque (ϵ is complex) — non-absorbing or transparent (ϵ is real)
- dispersive [$\epsilon = \epsilon(\omega)$][1] — non-dispersive ($\epsilon$ is independent of ω)
- nonlinear (ϵ depends on the fields $\boldsymbol{E}$ or $\boldsymbol{H}$) — linear (ϵ is independent of the fields)[2]

Besides the dielectric permittivity ϵ, the dielectric susceptibility χ is often introduced:

$$\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}. \tag{1.18}$$

If we substitute (1.18) into (1.16) we obtain the relation between ϵ and χ:

$$\epsilon = \epsilon_0 (1 + \chi). \tag{1.19}$$

Maxwell equations in the frequency domain

Let us emphasize that the ME are linear in linear media, *i.e.*, in the media where ϵ is independent of the fields. This property is of crucial importance because it allows introducing the superposition principle to the electromagnetic field theory; in other words we can solve the ME separately for each frequency component. The frequency components of the field vectors can be expressed as [see (1.3)]:

$$\boldsymbol{E}(t) = \boldsymbol{E}\, e^{i\omega t}\,; \tag{1.20}$$

analogous relations can be written also for $\boldsymbol{H}$, $\boldsymbol{D}$, and $\boldsymbol{B}$, and for the sources ρ and $\boldsymbol{j}$. The time differentiation of these quantities then simply leads to a multiplication by the angular frequency:

$$\frac{\partial \boldsymbol{E}(t)}{\partial t} = i\omega \boldsymbol{E}\, e^{i\omega t}\,.$$

If we put these relations into the ME we obtain:

$$\nabla \times \boldsymbol{E} + i\omega \boldsymbol{B} = 0\,, \tag{1.21a}$$
$$\nabla \times \boldsymbol{H} - i\omega \boldsymbol{D} = \boldsymbol{j}\,, \tag{1.21b}$$
$$\nabla \cdot \boldsymbol{D} = \rho\,, \tag{1.21c}$$
$$\nabla \cdot \boldsymbol{B} = 0\,. \tag{1.21d}$$

All the spectral components of the electromagnetic field should satisfy these equations supplemented by constitutive relations (1.16). Once these equations are solved for an arbitrary frequency component we can apply one of the expressions (1.6), (1.7) or (1.9) to find general polychromatic solutions.

[1] One intuitively understands (and we have already pointed it out) that the electromagnetic field can be decomposed into spectral components. The amplitudes of these components read as $\boldsymbol{E}(\omega)$, $\boldsymbol{D}(\omega)$, *etc.* In the case of a dispersive medium where $\epsilon = \epsilon(\omega)$, the equation (1.16) has to be understood as a relation between the spectral components of the electromagnetic field:

$$\boldsymbol{D}(\omega) = \epsilon(\omega) \boldsymbol{E}(\omega). \tag{1.16bis}$$

The phenomenon of the dispersion will be treated in detail in Chapter 2 of this textbook. The constitutive relations (1.16) and (1.16bis) will be further discussed in Chapter 6.

[2] The linear optics, treating cases where ϵ is independent of the fields, constitutes the principal subject of this textbook. The situations with $\epsilon = \epsilon(\boldsymbol{E})$ and/or $\epsilon = \epsilon(\boldsymbol{H})$ are treated within the so called nonlinear optics which is briefly introduced in Chapter 7.

B. Energy of the electromagnetic field, Poynting vector

It is well-known that any conservation law can be written in the following form:

$$\frac{\partial g}{\partial t} = -\nabla \cdot \boldsymbol{T} + P\,, \tag{1.22}$$

or using the Gauss' theorem

$$\int\limits_V \frac{\partial g}{\partial t}\, dV = -\oint\limits_S \boldsymbol{T} \cdot\, d\boldsymbol{S} + \int\limits_V P\, dV\,, \tag{1.23}$$

where g is the physical quantity to be conserved, $\boldsymbol{T}$ is its flux, and P is the density of its sources. V is an arbitrary volume and S is its delimiting closed surface. Both equations state that the increase of the quantity g in a volume V is equal to the amount of g which flows into the volume V plus the amount of g which is created in this volume.

We are looking for an analogous expression in the electromagnetism. The change of the electromagnetic energy stored in a volume V thus should be equal to the flux of the energy flowing into V reduced by the work that the field exerts on the charged particles in V. Point charges are accelerated due to the Lorentz force $q\,(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$; it then follows that the work done by the field per unit time is equal to: $q\boldsymbol{v}\cdot(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = q\boldsymbol{v}\cdot\boldsymbol{E}$. The magnetic field does not produce any work because the magnetic force is always perpendicular to the particle velocity $\boldsymbol{v}$. For a continuous distribution of the charges and of the current, the work of the field per unit time reads: $\boldsymbol{j}\cdot\boldsymbol{E}$; this term is known as the Joule heat and describes the energy dissipation. The energy conservation law then comes from (1.12b):

$$(\nabla\times\boldsymbol{H})\cdot\boldsymbol{E} - \frac{\partial \boldsymbol{D}}{\partial t}\cdot\boldsymbol{E} = \boldsymbol{j}\cdot\boldsymbol{E}$$

and from (1.12a):

$$(\nabla\times\boldsymbol{E})\cdot\boldsymbol{H} + \frac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{H} = 0\,.$$

Summing these two equations and taking into account the identity of the vectorial analysis

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H})$$

one obtains the following equation:

$$\frac{\partial U}{\partial t} + \nabla\cdot\boldsymbol{S} = -\boldsymbol{j}\cdot\boldsymbol{E} \tag{1.24}$$

with

$$\frac{\partial U}{\partial t} = \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{1.25}$$

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}\,. \tag{1.26}$$

For a linear and non-dispersive medium the equation (1.25) can be easily integrated[3] and one obtains:

$$U = \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}\right) \tag{1.27}$$

The terms of (1.24) have the dimension of the energy change per unit time in a unit volume. We have identified the right-hand-side term as an energy density loss (its transformation into the heat). The whole equation then can be interpreted as the conservation law of electromagnetic energy. U is thus interpreted as the electromagnetic energy density and $\boldsymbol{S}$, called Poynting vector, as its flux. The dimension of $\boldsymbol{S}$ is $Jm^{-2}s^{-1}$, its amplitude represents the radiation power passing through a unit surface in the direction of $\boldsymbol{S}$. Both these quantities are very important because they are directly measurable. Strictly speaking, the light detectors are not sufficiently rapid to be able to follow the fast oscillations at the optical frequencies ($\omega \approx 10^{15}s^{-1}$): they only measure mean values over a large number of periods of the electromagnetic wave:

$$\langle\boldsymbol{S}\rangle = \langle\boldsymbol{E}\times\boldsymbol{H}\rangle \tag{1.28}$$

$$\langle U\rangle = \frac{1}{2}\,\langle\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}\rangle \tag{1.29}$$

Using the complex formalism introduced on page 3 we can write in agreement with (1.11):

$$\langle\boldsymbol{S}\rangle = \frac{1}{2}\,\mathrm{Re}\{\boldsymbol{E}\times\boldsymbol{H}^*\}\,, \tag{1.30}$$

$$\langle U\rangle = \frac{1}{4}\,\mathrm{Re}\{\boldsymbol{E}\cdot\boldsymbol{D}^* + \boldsymbol{H}\cdot\boldsymbol{B}^*\}\,. \tag{1.31}$$

[3]The equation (1.25) cannot be integrated for the general case without the knowledge of the dispersion relation. In principle, the relation between $\boldsymbol{E}$ and $\boldsymbol{D}$ for a given medium should be substituted into (1.25), and only then the integration can be performed. Eqs. (2.38) and (2.39) show the result of such a calculation for a quasi-monochromatic wave in a dispersive medium.

2. Propagation in linear isotropic homogeneous media

2.1 Wave equation, plane monochromatic waves and their superpositions

A. Propagation in vacuum

The Maxwell equations along with the constitutive relations represent a set of coupled differential equations for four field vectors. It would be then convenient to derive equations which determine each of the field vectors separately. In the vacuum we state: $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\rho = 0$, $\boldsymbol{j} = 0$. We apply the ***curl*** operator ($\nabla\times$) to the equation (1.12a) and the operator $\partial/\partial t$ to (1.12b); comparing the two resulting relations we obtain:

$$\nabla \times (\nabla \times \boldsymbol{E}) + \epsilon_0\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0\,, \tag{2.1}$$

and, consequently, the wave equation for the electric field reads:

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0\,. \tag{2.2}$$

Similarly, one may find for the magnetic field:

$$\nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\,, \tag{2.3}$$

where

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}\,. \tag{2.4}$$

The solutions of the wave equation (2.2) are the plane monochromatic waves:

$$\boldsymbol{E} = \boldsymbol{E}_0\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\,, \tag{2.5}$$

the angular frequency ω and the wave vector $\boldsymbol{k}$ being connected by

$$|\boldsymbol{k}| \equiv k = \frac{\omega}{c}\,. \tag{2.6}$$

The wave front is perpendicular to $\boldsymbol{k}$ and it moves along the $\boldsymbol{k}$-direction with the speed of light c. If we substitute (2.5) into ME (1.12a) and (1.12b), we obtain:

$$\boldsymbol{H} = \boldsymbol{H}_0\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\,, \tag{2.7}$$

and

$$\boldsymbol{H}_0 = \eta_0^{-1}(\boldsymbol{s}\times\boldsymbol{E}_0)\,, \quad \boldsymbol{B}_0 = c^{-1}(\boldsymbol{s}\times\boldsymbol{E}_0)\,, \tag{2.8}$$

where $\boldsymbol{s} = \boldsymbol{k}/k$ is a unit vector parallel to $\boldsymbol{k}$. We thus obtain the following relation for the field amplitudes:

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv \eta_0\,, \quad \frac{E_0}{B_0} = \sqrt{\frac{1}{\mu_0\epsilon_0}} \equiv c\,, \tag{2.9}$$

($\eta_0 = 377\,\Omega$ is called the vacuum wave impedance) and — from (1.12c) and (1.12d) — a requirement of the transverse character of the electromagnetic field:

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = \boldsymbol{k}\cdot\boldsymbol{H}_0 = 0 \qquad (\boldsymbol{k}\perp\boldsymbol{E}_0\perp\boldsymbol{H}_0\perp\boldsymbol{k})\,. \tag{2.10}$$

The vectors $\boldsymbol{k}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ form a right-handed basis and it follows that $\boldsymbol{H}_0$ (and $\boldsymbol{B}_0$) are unambiguously determined by $\boldsymbol{k}$ and $\boldsymbol{E}_0$.

The independent solutions of the wave equations (2.2) and (2.3) are called modes of the electromagnetic radiation[1]. The modes must form a complete orthogonal basis in the space of all the solutions: any solution of the wave equation can be decomposed to a linear superposition of the modes:

$$\boldsymbol{E}(t,\boldsymbol{r}) = \iiint \boldsymbol{E}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, d\boldsymbol{k}\,, \tag{2.11a}$$

$$\boldsymbol{H}(t,\boldsymbol{r}) = \iiint \boldsymbol{H}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, d\boldsymbol{k}\,. \tag{2.11b}$$

[1]The plane waves do not represent the only possible system of solutions of the equations (2.2) and (2.3); these waves form a basis which is of great importance because the linear combinations correspond to the Fourier transform [see expression (2.12)]. On the other hand, the plane waves are not spatially limited: they extend to the infinity in the transverse direction. For this reason another system of solutions is often used in the laser physics: so called Gaussian beams. These modes are spatially limited: *i.e.*, they have a finite diameter and thus they are more suitable for the description of the laser beams. Other systems of solutions are used especially if specific boundary conditions are to be applied (*e.g.* Bessel functions in optical fibres *etc.*)

The amplitudes $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ thus stand for the spectral components (Fourier components) of a particular state of the radiation. If we need to describe *e.g.* a pulse of electromagnetic radiation propagating along the z-axis ($k_x = k_y = 0$) we can decompose it using the 1-dimensional Fourier transformation:

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(\omega)\, e^{i(\omega t - kz)}\, d\omega\,, \tag{2.12a}$$

$$\boldsymbol{H}(t,z) = \int \boldsymbol{H}_0(\omega)\, e^{i(\omega t - kz)}\, d\omega\,. \tag{2.12b}$$

Formally, this pulse can be characterized by a mean angular frequency ω_0, a mean wave vector $k_0 = \omega_0/c$, and a vectorial function $\boldsymbol{E}_1(t)$ describing its envelope:

$$\begin{aligned}\boldsymbol{E}(t,z) &= e^{i(\omega_0 t - k_0 z)} \int \boldsymbol{E}_0(\omega) e^{i((\omega-\omega_0)t - (k-k_0)z)}\, d\omega = \\ &= e^{i(\omega_0 t - k_0 z)} \int \boldsymbol{E}_0(\omega) e^{i((\omega-\omega_0)(t-z/c)}\, d\omega \;=\; \boldsymbol{E}_1(t - z/c)\, e^{i(\omega_0 t - k_0 z)}\,.\end{aligned} \tag{2.13}$$

This means that both the envelope and the phase of the pulse propagate with the speed of light c.

In principle, we can always take the plane harmonic waves as solutions of the wave equation; the Fourier transformation allows then acceding all possible physical solutions.

Let us go back to ME: if we are looking for the plane-wave solutions of ME we can directly replace the operator $\nabla\times$ by $-i\boldsymbol{k}\times$ and $\nabla\cdot$ by $-i\boldsymbol{k}\cdot$. Maxwell equations (1.21) then take the following form:

$$-i\boldsymbol{k}\times\boldsymbol{E} + i\omega\boldsymbol{B} = 0 \tag{2.14a}$$

$$-i\boldsymbol{k}\times\boldsymbol{H} - i\omega\boldsymbol{D} = \boldsymbol{j} \tag{2.14b}$$

$$-i\boldsymbol{k}\cdot\boldsymbol{D} = \rho \tag{2.14c}$$

$$-i\boldsymbol{k}\cdot\boldsymbol{B} = 0 \tag{2.14d}$$

The equations (2.14c) and (2.14d) mean that, in a medium without free charges ($\rho = 0$), the electromagnetic radiation (vectors $\boldsymbol{D}$ and $\boldsymbol{B}$) is always transverse.

B. Propagation in matter (with or without absorption)

We will consider a dielectric ($\epsilon \neq \epsilon_0$) non magnetic ($\mu = \mu_0$) medium. The current formed by the free charges is related to the electric field through the Ohm's law:

$$\boldsymbol{j} = \sigma\boldsymbol{E}\,, \tag{2.15}$$

where σ is the conductivity of the medium. The equation (1.21b) for the spectral component ω can be then rewritten in the following form:

$$\nabla\times\boldsymbol{H} = i\omega\left(\epsilon - \frac{i\sigma}{\omega}\right)\boldsymbol{E}\,. \tag{2.16}$$

It is clear that the effects of the current constituted by free charges can be simply included into a complex parameter $\hat{\epsilon} = \epsilon - i\sigma/\omega$. This parameter then accounts both for the dielectric contribution (bound charges) and for the ohmic effects (free charges) of the medium. Now let us examine eq. (1.12c):

$$\nabla\cdot(\epsilon\boldsymbol{E}) = \rho\,.$$

We can transform it as follows:

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \nabla \cdot (\epsilon - i\sigma/\omega)\boldsymbol{E} + i/\omega\, \nabla \cdot (\sigma \boldsymbol{E}) = \nabla \cdot (\hat{\epsilon}\boldsymbol{E}) - \nabla \cdot \boldsymbol{j}/(i\omega) = \rho\,.$$

Taking into account the equation of continuity (*i.e.*, conservation of the electric charges) $\nabla \cdot \boldsymbol{j} = -i\omega\rho$ one finds:

$$\nabla \cdot (\hat{\epsilon}\boldsymbol{E}) = 0\,.$$

Thus, we can—without loosing the general character of the solutions—omit the current density in the second ME (1.21b) and the density of free charges in the third ME (1.21c). We just need to postulate that all the effects related to both bound and free charges are described by a complex generalized dielectric permittivity ϵ (the "hat" will be omitted in the following text in order to simplify the notation). Let us emphasize that this treatment is valid only for the spectral components of the electromagnetic field (harmonic modes: $\boldsymbol{E}, \boldsymbol{H} \propto e^{i\omega t}$), for which we can replace the time derivative $\partial/\partial t$ by a multiplication by a factor of $i\omega$.

Now, we can follow step by step the procedure of solving the wave propagation in vacuum (pp. 11, 12). We find the wave equations:

$$\nabla^2 \boldsymbol{E} - \epsilon\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0\,, \qquad \nabla^2 \boldsymbol{H} - \epsilon\mu_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\,, \tag{2.17}$$

which, after substitution for plane wave solutions (2.5) and (2.7), become:

$$(k^2 - \epsilon\mu_0\omega^2) \begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{H}_0 \end{pmatrix} = 0\,. \tag{2.18}$$

One can introduce the complex refractive index $N = n - i\kappa$:[2]

$$N^2 = \epsilon\mu_0 c^2 \tag{2.19}$$

and the relative permittivity $\epsilon_r \equiv \varepsilon$ (called also frequently the dielectric constant):

$$\epsilon = \epsilon_0\epsilon_r \equiv \epsilon_0\varepsilon\,, \quad \varepsilon = \varepsilon' - i\varepsilon''\,. \tag{2.20}$$

In order to simplify the notation the symbol ε will be used throughout this textbook to denote the dimensionless relative permittivity while the symbol ϵ will keep the meaning of the permittivity expressed in $C\,V^{-1}\,m^{-1}$ following the definition (2.20). Taking into account (2.19) and (2.20), the relation between the refractive index and the dielectric constant reads:

$$N^2 = \varepsilon\,; \tag{2.21}$$

i.e., one obtains for the real and imaginary parts:

$$\begin{aligned} &\varepsilon' = n^2 - \kappa^2\,, && \varepsilon'' = 2n\kappa\,, \\ &n = \sqrt{\tfrac{1}{2}(\sqrt{\varepsilon'^2 + \varepsilon''^2} + \varepsilon')}\,, && \kappa = \sqrt{\tfrac{1}{2}(\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon')}\,. \end{aligned} \tag{2.22}$$

Eqs. (2.18) and (2.19) lead to the following dispersion relation:

$$k = \frac{\omega}{c} N\,, \tag{2.23}$$

[2] $n = \mathrm{Re}\{N\}$ and $\kappa = -\mathrm{Im}\{N\}$.

which implies that k must be complex. Consequently, the wave vector $\boldsymbol{k}$ must be a complex vector:

$$\boldsymbol{k} = \boldsymbol{k}' - i\boldsymbol{k}'' \,. \tag{2.24}$$

Following (2.17), (2.18), and (2.24):

$$k^2 = \boldsymbol{k} \cdot \boldsymbol{k} = k'^2 - k''^2 - 2i\boldsymbol{k}' \cdot \boldsymbol{k}'' \,, \tag{2.25}$$

where k' and k'' are the moduli of $\boldsymbol{k}'$ and $\boldsymbol{k}''$, respectively. The directions of the real and imaginary part of the wave vector ($\boldsymbol{k}'$ and $\boldsymbol{k}''$) can be in general different. They depend on the boundary conditions: the direction of the real part is perpendicular to the constant phase plane and the direction of the imaginary part is perpendicular to the constant amplitude plane:

$$\boldsymbol{E} = \boldsymbol{E}_0 \, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} = \boldsymbol{E}_0 \, e^{-\boldsymbol{k}''\cdot\boldsymbol{r}} \, e^{i(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r})} \,. \tag{2.26}$$

The boundary conditions are usually set at the input face of an absorbing sample. A detailed discussion of this problem will be provided later along with the analysis of the propagation through interfaces. Let us intuitively admit here that in the most frequent case, in which the electromagnetic wave propagates through the sample along the normal of the input face, $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are parallel to each other: $\boldsymbol{k}' \cdot \boldsymbol{k}'' = k'k''$. The dispersion relation then reads:

$$k = k' - ik'' = \frac{\omega}{c} N \tag{2.23bis}$$

with the real and imaginary part $k' = n\omega/c$, $k'' = \kappa\omega/c$.

It can be shown from ME that the field vectors are transverse in the case of $\boldsymbol{k}' \parallel \boldsymbol{k}''$:

$$\boldsymbol{k} \perp \boldsymbol{E}_0 \perp \boldsymbol{H}_0 \perp \boldsymbol{k} \,. \tag{2.27}$$

The wave impedance of the medium

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{2.28}$$

for non-magnetic media ($\mu = \mu_0$) is equal to:

$$\eta = \eta_0 / N \tag{2.29}$$

and one may derive from ME:

$$\frac{E_0}{H_0} = \eta \,, \tag{2.30}$$

$$\boldsymbol{H}_0 = \eta^{-1}(\boldsymbol{s} \times \boldsymbol{E}_0) \,, \quad \boldsymbol{B}_0 = N/c \, (\boldsymbol{s} \times \boldsymbol{E}_0) \,. \tag{2.31}$$

Again, one finds that the direction and the modulus of $\boldsymbol{H}_0$ are completely determined by the vectors $\boldsymbol{k}$ and $\boldsymbol{E}_0$ and by the dielectric and conductive properties of the medium (through ϵ or η).

Now let us calculate the Poynting vector, *i.e.*, the radiation intensity propagating in the medium:

$$\langle \boldsymbol{S} \rangle = \langle \boldsymbol{E} \times \boldsymbol{H} \rangle = \frac{1}{2} \, \mathrm{Re}\{ \boldsymbol{E} \times \boldsymbol{H}^* \} \,; \tag{2.32}$$

taking into account (1.21a) or (2.31):

$$\boldsymbol{H}_0 = \frac{1}{\omega\mu_0}(\boldsymbol{k}\times\boldsymbol{E}_0)\,, \tag{2.33}$$

one finds

$$\langle\boldsymbol{S}\rangle = \frac{1}{2\omega\mu_0}\,\mathrm{Re}\{\boldsymbol{E}\times(\boldsymbol{k}^*\times\boldsymbol{E}^*)\} = \frac{1}{2\omega\mu_0}\,\mathrm{Re}\{\boldsymbol{k}^*(\boldsymbol{E}\cdot\boldsymbol{E}^*) - (\underbrace{\boldsymbol{k}^*\cdot\boldsymbol{E}}_{0})\boldsymbol{E}^*\} =$$

$$= \frac{1}{2\omega\mu_0}\,\mathrm{Re}\{(\boldsymbol{k}' - i\boldsymbol{k}'')(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}\} = \frac{\boldsymbol{k}'}{2\omega\mu_0}|\boldsymbol{E}_0|^2 e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}\,.$$

In particular, if the wave propagates along z one obtains:

$$|\langle\boldsymbol{S}\rangle| = (2\omega\mu_0)^{-1}\,n\frac{\omega}{c}\,|\boldsymbol{E}_0|^2\,e^{-2\omega\kappa z/c} = \frac{n}{2\eta_0}\,|\boldsymbol{E}_0|^2\,e^{-\alpha z} \tag{2.34}$$

where the absorption coefficient α is defined by the following relation ($1/\alpha$ is called the penetration depth of the electromagnetic wave):

$$\alpha(\omega) = \frac{2\omega\kappa}{c}\quad\left(=\frac{4\pi\kappa}{\lambda}\right)\,. \tag{2.35}$$

The real part of the wave vector then determines the propagation direction of the electromagnetic energy; in an absorbing material, the power decreases exponentially with the thickness.

It is instructive to evaluate the rate of change of the electromagnetic energy in a dispersive medium. For a quasi-monochromatic wave

$$\boldsymbol{E}(t) = \boldsymbol{E}_0(t)\,e^{i\omega_0 t} = \int \boldsymbol{E}_\omega\,e^{i(\omega_0+\omega)t}\,d\omega\,, \tag{2.36}$$

(where E_0 is a slowly varying amplitude of the wave and the spectral components $E_\omega \neq 0$ only for $\omega \ll \omega_0$) the real dispersion in a material can be approximated by the first term of the Taylor power series in the vicinity of the carrier frequency ω_0 of the wave. We start from (1.25) where we need to evaluate the time derivative of $\boldsymbol{D}$. Omitting the vectorial notation of the fields we obtain [cf. also (6.3)]

$$\frac{\partial D(t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\epsilon(\omega_0+\omega)\,E_\omega e^{i(\omega_0+\omega)t}\,d\omega =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} i(\omega_0+\omega)\,\epsilon(\omega_0+\omega)\,E_\omega e^{i(\omega_0+\omega)t}\,d\omega \approx$$

$$\approx \frac{1}{2}\int_{-\infty}^{\infty}\left[i\omega_0\epsilon(\omega_0) + i\omega\frac{d[\omega_0\epsilon(\omega_0)]}{d\omega_0}\right]E_\omega e^{i(\omega_0+\omega)t}\,d\omega =$$

$$= i\omega_0\epsilon(\omega_0)E_0 e^{i\omega_0 t} + \frac{d[\omega_0\epsilon(\omega_0)]}{d\omega_0}\frac{dE_0}{dt}e^{i\omega_0 t}\,. \tag{2.37}$$

The mean value of the energy change is equal to

$$\left\langle\frac{\partial U}{\partial t}\right\rangle = \frac{1}{2}\mathrm{Re}\left[i\omega_0\epsilon|E_0|^2 + \frac{1}{2}\frac{d(\omega_0\epsilon)}{d\omega_0}\frac{\partial|E_0|^2}{\partial t} + \frac{1}{2}\mu_0\frac{\partial|H|^2}{\partial t}\right]\,. \tag{2.38}$$

Obviously, the first right-hand-side term has a dissipative character. The imaginary part of the permittivity ϵ'' thus accounts for the energy losses. The second and third terms describe the conservative part of the electromagnetic energy change; its dielectric part is related to the real part of the permittivity ϵ'. The conservative part of the energy then reads:

$$\langle U \rangle = \frac{1}{4}\left[\frac{d(\omega_0 \epsilon')}{d\omega_0}|E|^2 + \mu_0 |H|^2\right] . \tag{2.39}$$

C. Group velocity

Let us consider a laser pulse $\boldsymbol{E}(t,z)$ propagating along the z-axis in a non-absorbing (transparent) medium. If the amplitude of the pulse is only slowly varying in time, the spectral bandwidth of the pulse will be significantly smaller than its central frequency ($\Delta\omega \ll \omega_0$). Then we speak about a quasi-monochromatic wave.

The medium is characterized by the dispersion relation (2.23); we get for the real part (transparent medium):

$$k = \frac{\omega}{c} n(\omega) . \tag{2.40}$$

The refractive index n may depend on ω; consequently, k will also in principle depend on ω. And vice versa: if the wave vector k is regarded as the independent variable, ω may depend on k in a rather complicated way.

Our pulse is given by a linear combination of the eigenmodes:

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(k) e^{i(\omega(k)t - kz)}\, dk , \tag{2.41}$$

We can choose k_0 (or equivalently ω_0)

$$k_0 = \frac{\omega_0}{c}\, n(\omega_0) \tag{2.42}$$

such that $\boldsymbol{E}_0(k)$ has a maximum at k_0 and, typically, it is non-vanishing only in the close vicinity Δk of k_0 ($\Delta k \ll k_0$, quasi-monochromatic wave); ω_0 is called the central (mean) frequency of the pulse and k_0 is its central wave vector.

If we do not know exactly the variation of n versus ω, we can develop $\omega(k)$ in the Taylor series:

$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_0 (k - k_0) + \frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_0 (k - k_0)^2 \ \ldots . \tag{2.43}$$

In order to simplify the expressions we denote the coefficient $(d\omega/dk)_0$ as v_g and $(d^2\omega/dk^2)_0$ as β. Notice that v_g has the dimension of a velocity: it is called the group velocity and we will show its high importance for the pulse propagation. The equation (2.43) then reads:

$$\omega(k) = \omega_0 + v_g(k - k_0) + \frac{\beta}{2}(k - k_0)^2 \ \ldots . \tag{2.44}$$

Let us note that the pulse $\boldsymbol{E}(t,z)$ could have been also decomposed in spectral components of ω; one would then get an equation similar to (2.41):

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(\omega) e^{i(\omega t - k(\omega) z)}\, d\omega \tag{2.41bis}$$

with the Taylor expansion of $k(\omega)$:

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega}\right)_0 (\omega - \omega_0) + \frac{1}{2}\left(\frac{d^2k}{d\omega^2}\right)_0 (\omega - \omega_0)^2 \ \ldots\ , \tag{2.43bis}$$

which, taking into account (2.44), reads:

$$k(\omega) = k_0 + v_g^{-1}(\omega - \omega_0) + \frac{\psi}{2}(\omega - \omega_0)^2 \ \ldots\ . \tag{2.44bis}$$

The coefficient ψ is called the group dispersion parameter and it is related to β:

$$\psi = -\frac{\beta}{v_g^3}\,. \tag{2.45}$$

Let us first suppose that the spectrum of our pulse is narrow enough and that the dependence $\omega(k)$ is slow: we can then neglect the nonlinear terms of the expansion (2.44):

$$\boldsymbol{E}(t,z) = \int_{-\infty}^{\infty} \boldsymbol{E}_0(k) e^{i(\omega_0 t - k_0 z)} e^{i(v_g(k-k_0)t - (k-k_0)z)}\, dk = e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \boldsymbol{E}_0(k) e^{i(v_g t - z)(k-k_0)}\, dk$$

$$\boldsymbol{E}(t,z) = e^{i(\omega_0 t - k_0 z)}\, \boldsymbol{E}_1(z - v_g t) = \underbrace{e^{-ik_0(z-vt)}}_{(A)}\ \underbrace{\boldsymbol{E}_1(z - v_g t)}_{(B)}\,, \tag{2.46}$$

where the term (A) describes the field oscillation at the central frequency: the wavefront of this oscillation moves with the phase velocity

$$v = \frac{\omega_0}{k_0} = \frac{c}{n(\omega_0)}\,, \tag{2.47}$$

and the term (B) denotes the pulse envelope which conserves its shape during the propagation. Its propagation speed is just the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega\frac{dn}{d\omega}}\,. \tag{2.48}$$

This last formula was derived from the dispersion relation (2.40):

$$\begin{aligned} k &= \frac{\omega}{c} n(\omega) && / \ \frac{d}{dk} \\ \frac{dk}{dk} &= \frac{\omega}{c}\frac{dn(\omega)}{dk} + \frac{n(\omega)}{c}\frac{d\omega}{dk} \\ 1 &= \frac{\omega}{c}\frac{dn}{d\omega} v_g + \frac{n}{c} v_g \\ v_g &= \frac{c}{n + \omega dn/d\omega}\,. \end{aligned}$$

It follows from (2.34) and (2.46):

$$|\langle \boldsymbol{S}(z - v_g t)\rangle| \propto |\boldsymbol{E}_1(z - v_g t)|^2\,, \tag{2.49}$$

i.e., the pulse energy propagates with the group velocity v_g.

Spectral regions where the refractive index grows with frequency ($dn/d\omega > 0$) are called regions of normal dispersion; the ranges where $dn/d\omega < 0$ are referred to as ranges

of anomalous dispersion. Some important examples of the dispersion relations $n(\omega)$ will be discussed in the next section. For the moment, let us accept as a matter of fact that normal dispersion occurs in the regions of transparency of materials. The group velocity is smaller than the phase velocity in these ranges [*cf.* eqs. (2.47) and (2.48)]. In contrast, in the regions of anomalous dispersion, which are usually characterized by the absorption of the electromagnetic radiation, $dn/d\omega$ is negative: the group velocity is thus larger than the phase velocity. The group velocity can sometimes even exceed the vacuum speed of light c. However, it can be shown that this can happen only in the cases when the material absorption is large (the optical pulse is then strongly attenuated within fractions of wavelength) and/or in the cases when higher-order terms in eqs. (2.44) and (2.44bis) are not negligible for all spectral components of the pulse. These additional terms of the power expansion lead to a re-shaping of the optical pulse, namely to its temporal broadening (see the next paragraph): the group velocity then loses its clear physical sense that was demonstrated through (2.49) for the ideal case of linear dispersion. This means, for example, that for some particular cases of the dispersion relation and of the pulse shape some apparent velocities related to the pulse propagation (like *e.g.* the velocity of the motion of the peak intensity position of the pulse or the velocity of its temporal "center of gravity") can be greater than c. Nevertheless, this is not in contradiction with the theory of relativity and with the causality condition: in fact, it can be shown that the speed of the transmitted information (which is connected neither to the pulse maximum nor to its "center of gravity", but to the very first non-zero value of the pulse intensity) can never exceed c.

D. Group velocity dispersion

We have already mentioned that the quadratic terms β and ψ of the power expansions (2.44) and (2.44bis) lead to the temporal broadening of the pulse. This effect is known under the name of the group velocity dispersion and becomes very important especially for ultrashort pulses. Pulses with the pulse-length shorter than a fraction of picosecond have a broad spectrum (typically $\Delta k/k > 0.01$) and the nonlinear term $\beta(k-k_0)^2$ may significantly contribute if the pulse passes through a sufficiently thick dispersive sample. It is clear that the group velocity dispersion has to be carefully evaluated for example in the case of resonators for femtosecond lasers—where the pulses have to pass many times through all the dispersive elements without significant broadening—or in optical fibres—where the information is encoded into a sequence of optical pulses and has to be transmitted over many kilometers and still remain readable: the group velocity dispersion then limits the repetition rate of the pulses. The objective of the following calculation is to evaluate the temporal broadening of an optical pulse due to the propagation through a dispersive material.

Intuitive treatment The second order coefficient ψ of the power expansion (2.44bis) is connected to the frequency variation of the group velocity $dv_g/d\omega$:

$$\psi = \frac{d}{d\omega}\left(\frac{dk}{d\omega}\right) = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^2}\,\frac{dv_g}{d\omega}\,. \tag{2.50}$$

The expression (2.44bis) then reads:

$$k(\omega) = k_0 + \frac{1}{v_g}\,(\omega-\omega_0) - \frac{1}{2v_g^2}\,\frac{dv_g}{d\omega}\,(\omega-\omega_0)^2\ \ldots\,. \tag{2.51}$$

Comparison of the expressions (2.45) and (2.50) yields:

$$\beta = v_g \frac{dv_g}{d\omega} \,. \tag{2.52}$$

Let us consider a pulse with the spectral bandwidth $\Delta\omega$; the consequence of the non-vanishing group velocity dispersion is that two spectral components belonging respectively to the lower and upper edge of the pulse spectrum propagate with different velocities. The difference between these velocities is equal to:

$$\Delta v_g = \frac{dv_g}{d\omega} \, \Delta\omega \,.$$

The pulse broadening on a path-length L in the dispersive medium then can be estimated as:

$$\Delta\tau = \frac{L}{v_g^2} \, |\Delta v_g| = \frac{L}{v_g^2} \left| \frac{dv_g}{d\omega} \right| \, \Delta\omega = L \, |\psi| \, \Delta\omega \approx L \, |\psi| \, \frac{4 \ln 2}{\tau_0} \,, \tag{2.53}$$

where we have assumed that the original pulse length (*i.e.*, the pulse length before propagation through the dispersive medium) is related to the spectral bandwidth through the relation $\tau_0 = 4 \ln 2 / \Delta\omega$ (this relation assumes that the pulse is emitted by a coherent light source and that its shape can be approximated by a gaussian function).

One can express the parameter ψ in terms of the refractive index:

$$\frac{dk}{d\omega} = \frac{1}{c} \left(n + \omega \frac{dn}{d\omega} \right) = \frac{1}{c} \left(n - \lambda \frac{dn}{d\lambda} \right)$$

where we have used the relation $d\omega/\omega = -d\lambda/\lambda$; one obtains:

$$\psi \equiv \frac{d^2k}{d\omega^2} = \frac{1}{c} \frac{d}{d\omega} \left(n - \lambda \frac{dn}{d\lambda} \right) = -\frac{\lambda}{c\omega} \frac{d}{d\lambda} \left(n - \lambda \frac{dn}{d\lambda} \right) = \frac{\lambda^2}{c\omega} \frac{d^2n}{d\lambda^2} \,.$$

The equation (2.53) then reads:

$$\Delta\tau = L \frac{\lambda}{c} \frac{d^2n}{d\lambda^2} \, \Delta\lambda \,. \tag{2.54}$$

Rigorous treatment We start with a spectral decomposition of a pulse $\boldsymbol{E}(t,z)$:

$$\begin{aligned} \boldsymbol{E}(t,z) &= \int_{-\infty}^{\infty} \boldsymbol{E}_0(\omega) \, e^{i(\omega t - k(\omega) z)} \, d\omega \\ &= e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \boldsymbol{E}_0(\omega) e^{i(\omega-\omega_0)(t - z/v_g)} \, e^{-i\psi z/2 \, (\omega-\omega_0)^2} \, d\omega \,. \end{aligned} \tag{2.55}$$

Let us define:

$$T = t - z/v_g \,, \qquad\qquad \xi = -\psi z/2 \,,$$

to obtain

$$\boldsymbol{E}(t,z) = e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \boldsymbol{E}_0(\omega + \omega_0) \, e^{iT\omega} \, e^{i\xi\omega^2} \, d\omega \,.$$

The Fourier transform of a convolution of two functions is equal to the product of their respective Fourier transforms (see Appendix A):

$$FT(f * g) = 2\pi FT(f) \cdot FT(g)\,.$$

In our case we take:

$$FT(f) = \boldsymbol{E}_0(\omega + \omega_0)\,, \quad f = \boldsymbol{E}_1(T) = \boldsymbol{E}_1(t - z/v_g) \quad \text{[see expressions (2.13) and (2.46)]};$$
$$FT(g) = e^{i\xi\omega^2}\,;$$

Calculation of g:

$$\int_{-\infty}^{\infty} e^{iT\omega}\, e^{i\xi\omega^2}\, d\omega = \int_{-\infty}^{\infty} e^{i\xi(\omega^2 + T\omega/\xi)}\, d\omega = e^{-i\frac{T^2}{4\xi}} \int_{-\infty}^{\infty} e^{i\xi(\omega + T/(2\xi))^2}\, d\omega =$$
$$= e^{-i\frac{T^2}{4\xi}} \int_{-\infty}^{\infty} e^{i\xi y^2}\, dy = \sqrt{\frac{i\pi}{\xi}} e^{-i\frac{T^2}{4\xi}}\,,$$

where we have used the identity

$$\int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy = \sqrt{\frac{\pi}{\alpha}}$$

which is valid for $\mathrm{Re}\{\alpha\} \geq 0$. Finally, we obtain:

$$\boldsymbol{E}(t, z) = \frac{e^{i(\omega_0 t - k_0 z)}}{2\pi} \sqrt{\frac{i\pi}{\xi}} \int_{-\infty}^{\infty} \boldsymbol{E}_1(\tau)\, e^{-i\frac{(\tau - T)^2}{4\xi}}\, d\tau\,;$$

Notice that the pulse envelope depends not only on $T = t - z/v_g$ but also explicitly on z through the parameter $\xi = -\psi z/2$. This means that the pulse profile changes along z. The pre-factor $1/\sqrt{\xi}$ describes a decrease of the pulse peak intensity with z. On the other hand, the total pulse energy should be conserved (we consider a non-absorbing medium): this means that the pulse length should increase with z.

In order to be able to draw quantitative conclusions we will assume a specific temporal profile of the pulse:

$$\boldsymbol{E}_1(t, z = 0) = A\, e^{-\alpha t^2}\,;$$

the pulse hits the input face of the dispersive medium in the plane $z = 0$, its length (*i.e.*, the temporal distance between the times where the instantaneous intensity is equal to one half of the maximum intensity: this quantity is often called full-width-at-half-maximum or FWHM) is equal to

$$\tau_0 = \sqrt{\frac{2\ln 2}{\alpha}}\,.$$

We have to calculate the pulse FWHM $\tau(L)$ after having passed through the dispersive sample of thickness L.

$$\int_{-\infty}^{\infty} e^{-\alpha\tau^2}\, e^{-i\frac{(\tau-T)^2}{4\xi}}\, d\tau = e^{-i\frac{T^2}{4\xi}} \int_{-\infty}^{\infty} e^{-(\alpha+\frac{i}{4\xi})\tau^2+\frac{i}{2\xi}\tau T}\, d\tau =$$

$$= e^{-i\frac{T^2}{4\xi}}\, e^{-\frac{T^2}{4\xi(4\xi\alpha+i)}} \int_{-\infty}^{\infty} e^{-(\alpha+\frac{i}{4\xi})\,(\tau-\frac{iT}{4\xi\alpha+i})^2}\, d\tau = e^{-i\frac{T^2}{4\xi}}\, e^{-\frac{T^2}{4\xi(4\xi\alpha+i)}} \sqrt{\frac{4\pi\xi}{4\xi\alpha+i}} =$$

$$= \sqrt{\frac{4\pi\xi}{4\xi\alpha+i}}\; e^{-\frac{\alpha T^2}{(4\xi\alpha)^2+1}}\, e^{-i\frac{\alpha T^2\, 4\xi\alpha}{(4\xi\alpha)^2+1}}\,.$$

$$\boldsymbol{E}(t,z) = A\, e^{i(\omega_0 t - k_0 z)}\, \frac{1}{\sqrt{1-i\delta}}\, e^{-\alpha T^2 \frac{1+i\delta}{1+\delta^2}}\,, \tag{2.56}$$

with $\delta = 4\xi\alpha = -2\psi z\alpha$. The pulse length at the end of the dispersive medium ($z = L$) is then equal to:

$$\tau(L) = \tau_0\sqrt{1+\delta^2} = \tau_0\sqrt{1+\left(\frac{4\ln 2 L\psi}{\tau_0^2}\right)^2}\,. \tag{2.57}$$

If the distance L is very long one can neglect the initial pulse length compared to the final one. One then finds:

$$\tau(L) = L\psi\,\frac{4\ln 2}{\tau_0}\,. \tag{2.58}$$

This last equation is identical to (2.53) which was obtain using intuitive arguments.

Eq. (2.56) allows evaluation of the variation of the phase $\Phi(t,z)$ of the electric field and namely of the "local" frequency $\omega(t,z)$ of the field oscillations. Indeed, this frequency appears now different for different places of the pulse:

$$\omega(t,z) = \frac{\partial\phi}{\partial t} = \omega_0 + \frac{\partial}{\partial t}\left(-\frac{\alpha T^2\delta}{1+\delta^2}\right) = \omega_0 - \frac{2\alpha(t-z/v_g)\,\delta}{1+\delta^2}\,. \tag{2.59}$$

The local frequency varies linearly about the mean frequency ω_0. This effect is called "chirp"; also the parameter δ is usually referred to as the chirp parameter. In the case of a positive group velocity dispersion ($\psi \equiv d^2k/d\omega^2 > 0$) the trailing edge of the pulse will have a larger local frequency than the leading edge.

The propagation of an ultrashort pulse in vacuum and in dispersive media is shown in Fig. 2.1. If the central wavelength of this pulse were about 800 nm (typical wavelength of pulses emitted by Ti:sapphire femtosecond lasers) the FWHM (τ_0) of the pulse corresponding to the number of cycles in Fig. 2.1 would be of about 10^{-14} s.

E. Exercises

1. Absorption of the electromagnetic energy

Let us consider a monochromatic wave

$$\mathrm{Re}\{\boldsymbol{E}_0\, e^{i\omega t}\}$$

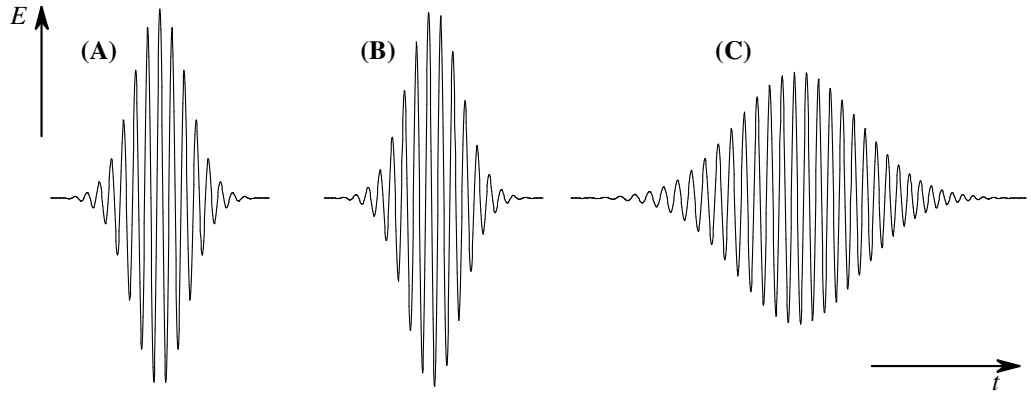

Figure 2.1: Example of the propagation of an ultrashort pulse. (A) Original pulse; (B) pulse after a propagation through a medium with $v \neq v_g$ but without the group velocity dispersion (note that the field fringes are shifted with respect to the pulse envelope); (C) pulse after a propagation through a medium with a non-vanishing group velocity dispersion.

propagating in an absorbing medium. We know that in a non-absorbing medium the quantity

$$\left\langle \boldsymbol{E}\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}\frac{\partial \boldsymbol{B}}{\partial t} \right\rangle$$

corresponds to the rate of change of the electromagnetic energy density in that medium. The energy carried by a monochromatic wave per unit of time should be constant due to the stationary nature of this wave.

Calculate the rate of change of the electromagnetic energy for a monochromatic wave in an absorbing medium characterized by a complex permittivity. If this rate does not vanish it must correspond to the dissipation of the electromagnetic energy, *i.e.*, to the Joule heat produced per unit of time and per unit of volume. Show that this heat is proportional to the imaginary part of the permittivity. [Note that by setting an equivalence between free and bound charges (2.16) the Joule heat is no more accounted for by $-\boldsymbol{j}\cdot\boldsymbol{E}$ as in (1.24) but it must contribute to the product $\boldsymbol{E}\cdot\boldsymbol{D}$]

2. Poynting vector in an absorbing medium

Let us consider a monochromatic wave of the preceding exercise; let it be a plane wave propagating along z. Write the amplitude coefficient $\boldsymbol{E}_0$ as a function of z and of the optical constants n and κ. Calculate the mean value of the Poynting vector and show that it exhibits an exponential decrease with z. This decrease indicates the dissipation (absorption) of the electromagnetic energy. Check that the equation

$$\left\langle \boldsymbol{E}\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}\frac{\partial \boldsymbol{B}}{\partial t} \right\rangle + \nabla\cdot\langle \boldsymbol{S}\rangle = 0$$

holds. This brings another proof of the interpretation of the preceding exercise, *i.e.*, that its first term corresponds to the rate of produced heat.

3. Sum of two plane harmonic waves: group velocity, beats

We have two monochromatic waves with angular frequencies ω_1 and ω_2. These frequencies are different but close to each other: $\omega_1, \omega_2 \gg |\omega_1 - \omega_2|$.

$$E_1 = A\cos(\omega_1 t - k_1 z)$$
$$E_2 = A\cos(\omega_2 t - k_2 z)\,.$$

Find the expression for the total field given by the superposition of the two waves. Show that we can interpret it as a single wave with an angular frequency $(\omega_1+\omega_2)/2$ and with a time-modulated amplitude (beats). Calculate the frequency and the propagation velocity of this modulation. Make a connection with the group velocity.

4. Group velocity
We have the dispersion curves $n(\omega)$ and $n(\lambda)$; find the expression for the group velocity, *i.e.*, calculate $v_g = v_g(n, \omega)$ and $v_g = v_g(n, \lambda)$.

5. Spectral decomposition of a gaussian pulse
Gaussian pulse at $z = 0$:

$$E(t) = A\, e^{i\omega_0 t}\, e^{-\alpha t^2}.$$

Calculate the pulse length τ_0 (FWHM = full-width-at-half-maximum), its spectrum [*i.e.*, spectral components of the amplitude $E_0(\omega)$ and of the intensity $I(\omega)$], the FWHM of the spectrum $\Delta\omega$ and $\Delta\lambda$.

Consider short gaussian pulses at 800 nm (Ti:sapphire laser) with τ_0 = (a) 10 fs, (b) 100 fs, (c) 10 ps (1 fs = 10^{-15} s, 1 ps = 10^{-12} s); find $\Delta\lambda$ for these three cases. The dependence is non-linear in λ: find the spectral FWHM for $\lambda = 400$ nm and $\tau_0 = 10$ fs.

6. Propagation of a gaussian pulse
We propagate the pulse of the exercise 5 through a medium. Express the field $E(t, z)$ as a sum of spectral components $E(\omega)$. Evaluate this integral assuming:

$$k = k_0 + (\omega - \omega_0)v_g^{-1}\,;$$

show the phase and group velocity in the final expression.

7. Group velocity dispersion
Consider the power expansions (2.43) and (2.43bis). Their last terms, $d^2\omega/dk^2$ and $d^2k/d\omega^2$, are responsible for the phase modulation and consequently for the pulse spread. We dispose of the dispersion curve $n(\lambda)$; express the coefficients $d^2\omega/dk^2$ and $d^2k/d\omega^2$.

8. Temporal broadening of a pulse
A gaussian pulse considered in exercise 5 passes through a glass plate. Calculate the pulse length at the output face of the plate assuming that $n(\lambda)$ is known. Show that the second derivative of a function $f(x)$ can be approximated by:

$$\frac{d^2 f}{dx^2} \approx \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{(\Delta x)^2}\,.$$

Central wavelength of a pulse is 800 nm (Ti:sapphire laser); compare the temporal broadening of two pulses with the initial lengths 10 and 100 fs propagating through a glass plate with the thickness: (a) 3 mm, (b) 10 mm. Refractive index of the glass (BK7): $n(750\,\text{nm}) = 1.51141$, $n(800\,\text{nm}) = 1.51035$, $n(850\,\text{nm}) = 1.50941$.