Lecture 8: Light propagation in anisotropic media

Petr Kužel

- Tensors — classification of anisotropic media
- Wave equation
- Eigenmodes — polarization eigenstates
- Normal surface (surface of refractive indices)
- Indicatrix (ellipsoid of refractive indices)

Anisotropic response

Isotropic medium, linear response:

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 (1 + \chi) \boldsymbol{E}$$

Anisotropic medium: the polarization direction does not coincide with the field direction:

$$P_x = \varepsilon_0 (\chi_{11} E_x + \chi_{12} E_y + \chi_{13} E_z),$$
$$P_y = \varepsilon_0 (\chi_{21} E_x + \chi_{22} E_y + \chi_{23} E_z),$$
$$P_z = \varepsilon_0 (\chi_{31} E_x + \chi_{32} E_y + \chi_{33} E_z).$$

⇨ Tensors

$$P_i = \varepsilon_0 \chi_{ij} E_j$$
$$D_i = \varepsilon_{ij} E_j = \varepsilon_0 (1 + \chi_{ij}) E_j$$

$\left.\begin{array}{l} \boldsymbol{k} \perp \boldsymbol{D} \\ \boldsymbol{S} \perp \boldsymbol{E} \end{array}\right\}$ ***k*** is not parallel to ***S***

Tensors

Transformation of a basis:

$$\boldsymbol{f}_j = A_{ij}\boldsymbol{e}_i$$

Corresponding transformation of a tensor:

$x_k = A_{ki}x'_i$ $\quad$ $x'_i = A^{-1}_{ik}x_k$ $\quad$ 1st rank (vector)

$t'_{kl} = A^{-1}_{ik}A^{-1}_{jl}t_{ij}$ $\quad$ $t_{ij} = A_{ki}A_{lj}t'_{kl}$ $\quad$ 2nd rank

$p_{i_1,i_2\ldots i_n} = A_{k_1i_1}A_{k_2i_2}\ldots A_{k_ni_n}p'_{k_1,k_2\ldots k_n}$ $\quad$ nth rank

Tensors: symmetry considerations

Intrinsic symmetry: reflects the character of the physical phenomenon represented by the tensor (usually related to the energetic considerations)

$$\varepsilon_{ik} = \varepsilon^{*}_{ki}$$

Extrinsic symmetry: reflects the symmetry of the medium

$$p_{i_1,i_2 \ldots i_n} = A_{k_1 i_1} A_{k_2 i_2} \ldots A_{k_n i_n} p_{k_1,k_2 \ldots k_n}$$

$$\varepsilon_{ij} = A_{ki} A_{lj} \, \varepsilon_{kl}$$

Example: two-fold axis along *z*:

$$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varepsilon_{13} = A_{11} A_{33} \varepsilon_{13} = -\varepsilon_{13} \quad \Rightarrow \quad \varepsilon_{13} = \varepsilon_{31} = 0$$

$$\varepsilon_{23} = A_{22} A_{33} \varepsilon_{23} = -\varepsilon_{23} \quad \Rightarrow \quad \varepsilon_{23} = \varepsilon_{32} = 0$$

Tensors: symmetry considerations

Second example: higher order axis along *z*:

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varepsilon_{11} = c^2\varepsilon_{11} + s^2\varepsilon_{22} - 2sc\varepsilon_{12}$$
$$\varepsilon_{22} = s^2\varepsilon_{11} + c^2\varepsilon_{22} + 2sc\varepsilon_{12}$$
$$\varepsilon_{12} = sc\varepsilon_{11} - sc\varepsilon_{22} + \left(c^2 - s^2\right)\varepsilon_{12}$$

The above system of equations leads to:

$$(\varepsilon_{11} - \varepsilon_{22})s^2 = -2sc\varepsilon_{12}$$
$$(\varepsilon_{11} - \varepsilon_{22})sc = 2s^2\varepsilon_{12}$$

Solution for a higher (than 2) order axis: $\sin\alpha \neq 0$

$$\varepsilon_{11} = \varepsilon_{22}, \qquad \varepsilon_{12} = 0$$

ε_{ij}

Axes systems:

- crystallographic
- optical (dielectric)
- laboratory

Optical symmetry	Crystallographic system	Dielectric tensor	
		optical system of axes	crystallographic system of axes
isotropic	cubic	$\varepsilon = \varepsilon_0 \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix}$	$\varepsilon = \varepsilon_0 \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix}$
uniaxial	hexagonal tetragonal trigonal	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}$	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}$
biaxial	orthorhombic	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$
	monoclinic	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$	$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}$
	triclinic	$\varepsilon = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$	$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}$

Wave equation

Maxwell equations for harmonic plane waves:

$$\boldsymbol{k} \wedge \boldsymbol{E} = \omega\mu_0 \boldsymbol{H}$$

$$\boldsymbol{k} \wedge \boldsymbol{H} = -\omega\varepsilon \cdot \boldsymbol{E}$$

in the system of principal optical axes

$$\varepsilon = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$$

Wave equation:

$$\boldsymbol{k} \wedge (\boldsymbol{k} \wedge \boldsymbol{E}) + \omega^2 \mu_0 \varepsilon \cdot \boldsymbol{E} \quad = \quad \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}) - k^2 \boldsymbol{E} + \omega^2 \mu_0 \varepsilon \cdot \boldsymbol{E} = 0$$

or

$$\boldsymbol{s}(\boldsymbol{s} \cdot \boldsymbol{E}) - \boldsymbol{E} + \frac{\omega^2}{k^2 c^2} \varepsilon_r \cdot \boldsymbol{E} = 0$$

Wave equation: continued

We define effective refractive index:

$$n = \frac{kc}{\omega}$$

Wave equation in the matrix form:

$$\begin{pmatrix} n_x^2 - n^2\left(s_y^2 + s_z^2\right) & n^2 s_x s_y & n^2 s_x s_z \\ n^2 s_x s_y & n_y^2 - n^2\left(s_x^2 + s_z^2\right) & n^2 s_y s_z \\ n^2 s_x s_z & n^2 s_y s_z & n_z^2 - n^2\left(s_x^2 + s_y^2\right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Solutions of this homogeneous system: **det = 0**.

- Effective refractive index *n* for a given propagation direction
- Frequency ω for a given wave vector ***k***
- Surface of accepted wave vector moduli for a given ω (normal surface or surface of refractive indices)

Wave equation: continued

After having developed the determinant one obtains:

$$\left(\frac{\omega^2 n_x^2}{c^2}-k^2\right)\left(\frac{\omega^2 n_y^2}{c^2}-k^2\right)\left(\frac{\omega^2 n_z^2}{c^2}-k^2\right)+k_x^2\left(\frac{\omega^2 n_y^2}{c^2}-k^2\right)\left(\frac{\omega^2 n_z^2}{c^2}-k^2\right)+$$

$$k_y^2\left(\frac{\omega^2 n_x^2}{c^2}-k^2\right)\left(\frac{\omega^2 n_z^2}{c^2}-k^2\right)+k_z^2\left(\frac{\omega^2 n_x^2}{c^2}-k^2\right)\left(\frac{\omega^2 n_y^2}{c^2}-k^2\right)=0\,.$$

or in terms of the effective refractive index:

$$\left(n_x^2-n^2\right)\left(n_y^2-n^2\right)\left(n_z^2-n^2\right)+$$

$$n^2\left[s_x^2\left(n_y^2-n^2\right)\left(n_z^2-n^2\right)+s_y^2\left(n_x^2-n^2\right)\left(n_z^2-n^2\right)+s_z^2\left(n_x^2-n^2\right)\left(n_y^2-n^2\right)\right]=0\,.$$

This is a quadratic equation in n^2, thus it provides two eigenmodes for a given direction of propagation $\boldsymbol{s}$.

Wave equation: continued

In the case when $n \neq n_i$, the wave equation can be further simplified:

$$\frac{s_x^2}{n^2 - n_x^2} + \frac{s_y^2}{n^2 - n_y^2} + \frac{s_z^2}{n^2 - n_z^2} = \frac{1}{n^2}$$

Polarization of the electric field for the eigenmodes:

$$\begin{pmatrix} n_x^2 - n^2\left(s_y^2 + s_z^2\right) & n^2 s_x s_y & n^2 s_x s_z \\ n^2 s_x s_y & n_y^2 - n^2\left(s_x^2 + s_z^2\right) & n^2 s_y s_z \\ n^2 s_x s_z & n^2 s_y s_z & n_z^2 - n^2\left(s_x^2 + s_y^2\right) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

For $n \neq n_i$: $\quad \boldsymbol{E} = \begin{pmatrix} \dfrac{s_x}{N^2 - n_x^2} \\ \dfrac{s_y}{N^2 - n_y^2} \\ \dfrac{s_z}{N^2 - n_z^2} \end{pmatrix}$

Isotropic case

$$n_x = n_y = n_z \equiv n_0$$

wave equation:

$$\left(n_0^2 - n^2\right)^2 n^2 = 0$$

Normal surface: degenerated sphere

Eigen-polarization: arbitrary

(Isotropic materials, cubic crystals)

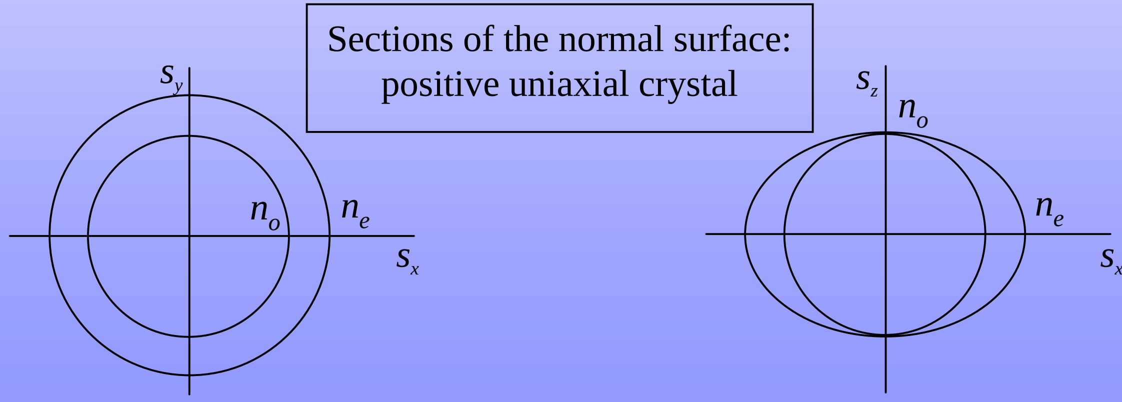
Uniaxial case

$n_x = n_y \equiv n_o$ and $n_z \equiv n_e \neq n_o$

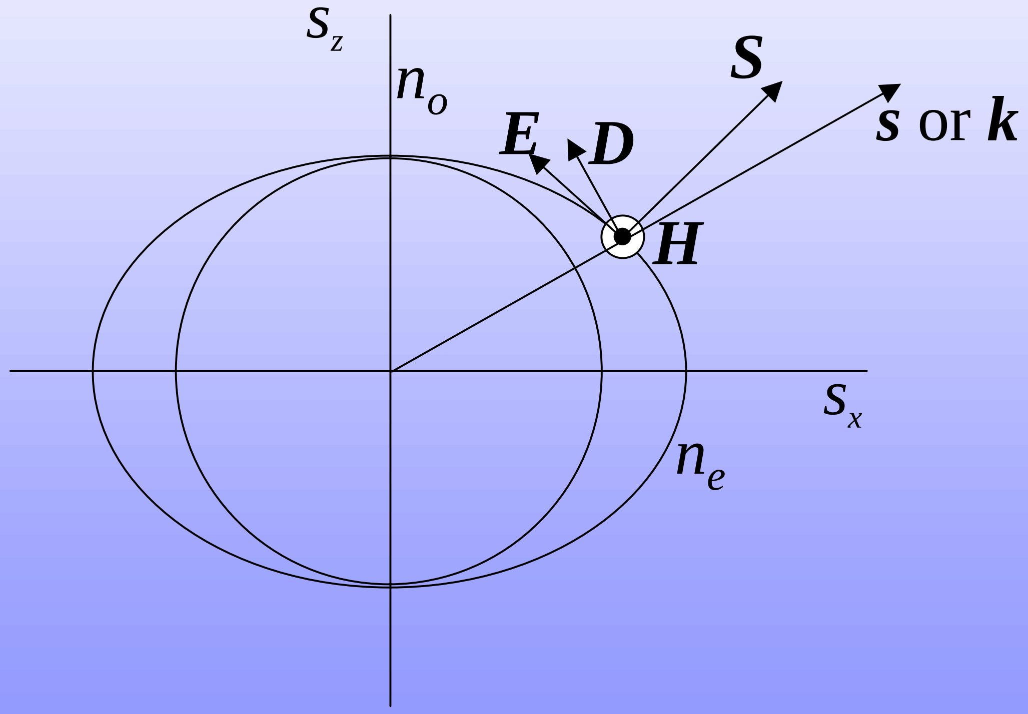
wave equation:

$$\left(n_o^2 - n^2\right)\left(\frac{1}{n^2} - \frac{s_x^2 + s_y^2}{n_e^2} - \frac{s_z^2}{n_o^2}\right) = 0$$

Normal surface: sphere + ellipsoid with one common point in the *z*-direction



Uniaxial case: continued



Uniaxial case: continued

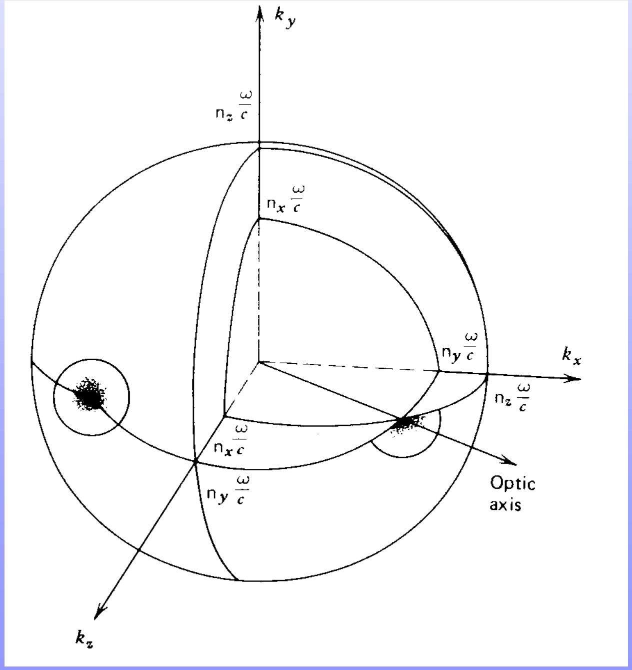
Propagation ($z \equiv$ optical axis)	ordinary ray	extraordinary ray
$\boldsymbol{k} \,//\, z$	degenerated case (equivalent to isotropic medium); index n_o, $\boldsymbol{E} \perp \boldsymbol{k}$	
$\boldsymbol{k} \perp z$	index n_o, $\boldsymbol{E}$ in the (xy) plane, $\boldsymbol{E} \perp \boldsymbol{k}$	index n_e, $\boldsymbol{E} \,//\, z$
$\boldsymbol{k}$: angle θ with z		index $n(\theta)$, $\boldsymbol{E}$ in the $(\boldsymbol{k}z)$ plane, $\boldsymbol{E} \cdot \boldsymbol{k} \neq 0$

with $$\frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2} = \frac{1}{n^2(\theta)}$$

Biaxial case

$n_x < n_y < n_z$

Normal surface has a complicated form

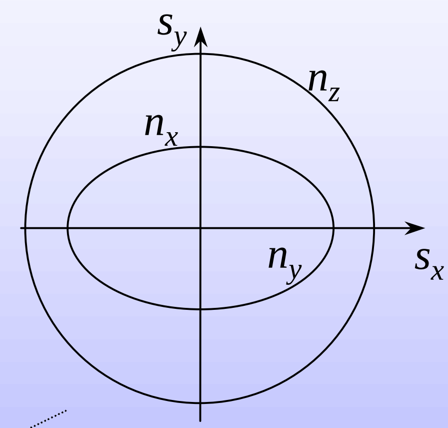


Biaxial case: continued

Sections of the normal surface by the planes *xy*, *xz*, and *yz*:

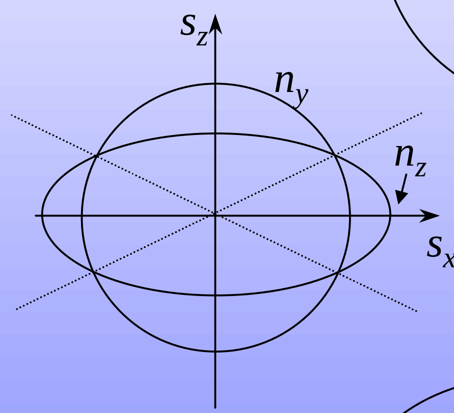
$s_x = 0$:

$$\left(\frac{1}{n^2}-\frac{s_z^2+s_y^2}{n_x^2}\right)\left(\frac{1}{n^2}-\frac{s_z^2}{n_y^2}-\frac{s_y^2}{n_z^2}\right)=0$$



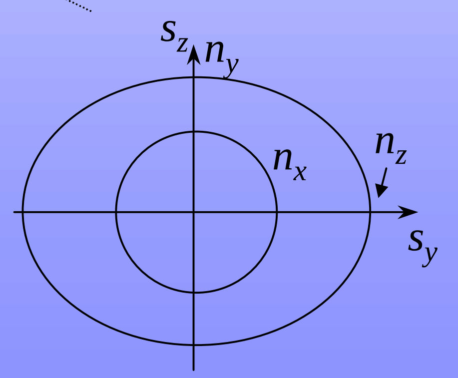
$s_y = 0$:

$$\left(\frac{1}{n^2}-\frac{s_x^2+s_z^2}{n_y^2}\right)\left(\frac{1}{n^2}-\frac{s_x^2}{n_z^2}-\frac{s_z^2}{n_x^2}\right)=0$$



$s_z = 0$:

$$\left(\frac{1}{n^2}-\frac{s_x^2+s_y^2}{n_z^2}\right)\left(\frac{1}{n^2}-\frac{s_x^2}{n_y^2}-\frac{s_y^2}{n_x^2}\right)=0$$



Group velocity in anisotropic media

Maxwell equations in the Fourier space:

$$\boldsymbol{k} \wedge \boldsymbol{E} = \omega\mu_0 \boldsymbol{H} \qquad /\cdot \boldsymbol{H}$$

$$\boldsymbol{k} \wedge \boldsymbol{H} = -\omega\varepsilon \cdot \boldsymbol{E} \qquad /\cdot \boldsymbol{E}$$

One can finally obtain:

$$\omega = \frac{2\boldsymbol{k} \cdot \left(\boldsymbol{E} \wedge \boldsymbol{H}\right)}{\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}} = \boldsymbol{k} \cdot \frac{\boldsymbol{S}}{U}$$

This means:

$$\boldsymbol{v}_g = \nabla_{\boldsymbol{k}} \omega = \frac{\boldsymbol{S}}{U} = \boldsymbol{v}_e$$

Birefringence

Isotropic medium: incident plane wave, reflected plane wave

Anisotropic medium: refracted wave is decomposed into the eigenmodes

- Tangential components should be conserved on the interface:

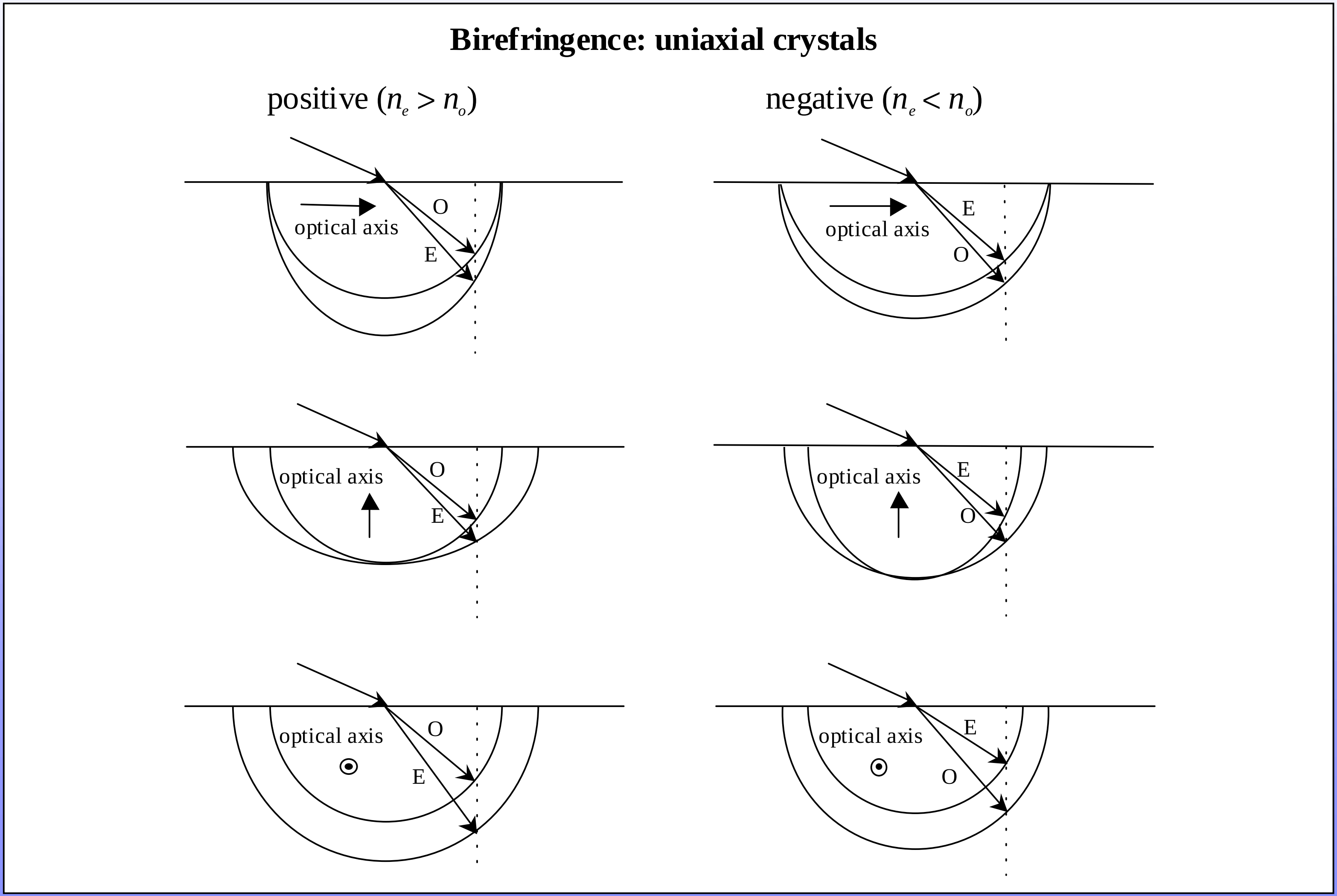
$$k_i \sin \alpha_i = k_1 \sin \alpha_1 = k_2 \sin \alpha_2$$

- $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are not constant but depend on the propagation direction
- Thus we get generalized Snell's law for an ordinary and an extra-ordinary beam:

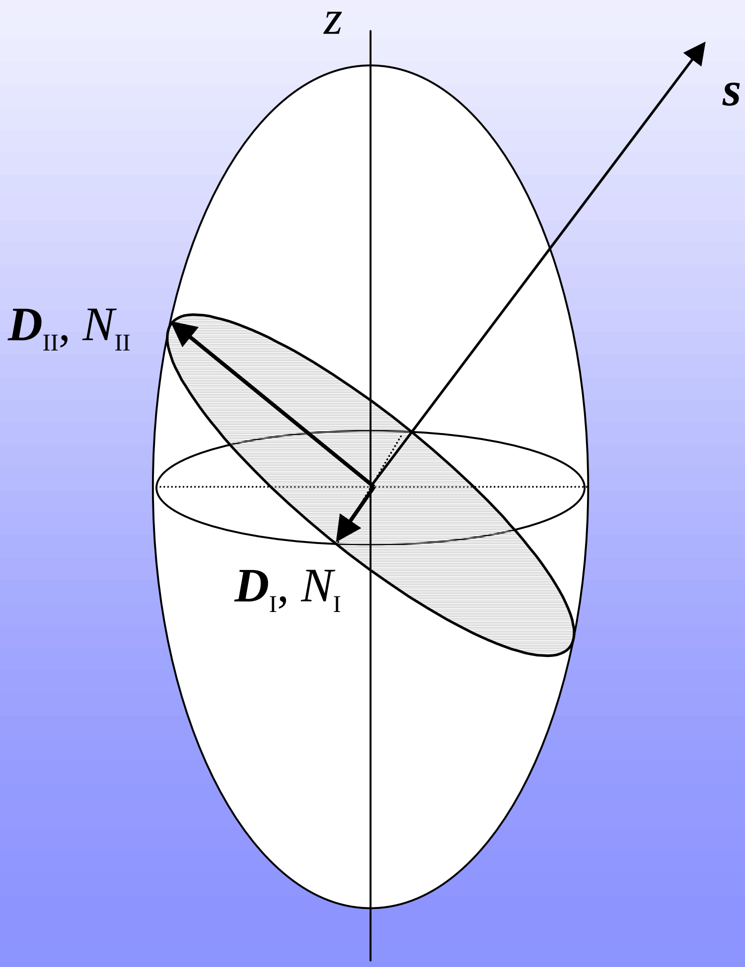
$$n_i \sin \alpha_i = n_o \sin \alpha_1 = n(\alpha_2) \sin \alpha_2$$

- Analytic solution only for special cases
- Numerical solution
- Graphic solution

Birefringence: graphic solution



Indicatrix: ellipsoid of indices



$$\frac{x^2}{n_x^2}+\frac{y^2}{n_y^2}+\frac{z^2}{n_z^2}=1$$