Layered structures: transfer matrix formalism

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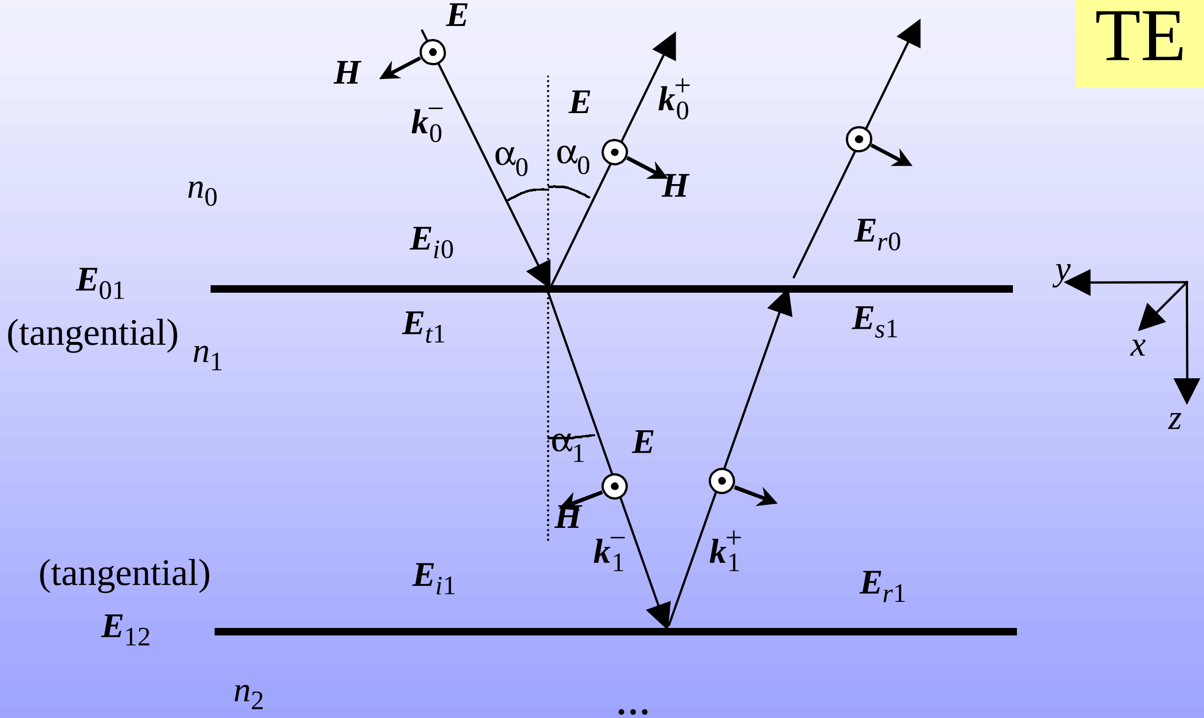
- Interfaces between LHI media
- Transfer matrix formalism
    - Practically only one formula is to be known in order to calculate any structure
- Applications:
    - Antireflective coatings
    - Dielectric mirrors, Chirped mirrors
    - Laser output couplers
    - Beam-splitters
    - Beam-splitting mirrors
    - Interference filters

Transfer matrix formalism

TE polarization



$$E_{01} = E_{t1} + E_{s1}$$
$$\eta_0 H_{01} = (E_{t1} - E_{s1})\gamma$$

$$E_{12} = E_{i1} + E_{r1}$$
$$\eta_0 H_{12} = (E_{i1} - E_{r1})\gamma$$

$$\left(\begin{array}{c} \gamma = n_1 \cos\alpha_1 \\ \delta = d_1 k_{z1} = \dfrac{2\pi}{\lambda} n_1 d_1 \cos\alpha_1 \end{array} \right)$$

$$E_{i1} = E_{t1} e^{-i\delta}$$
$$E_{s1} = E_{r1} e^{-i\delta}$$

Introduction of transfer matrix

$$\begin{pmatrix} E_{01} \\ \eta_0 H_{01} \end{pmatrix} = \begin{pmatrix} \cos\delta & \dfrac{i\sin\delta}{\gamma} \\ i\gamma\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} E_{12} \\ \eta_0 H_{12} \end{pmatrix}$$

Transfer matrix connects tangential fields on both ends of a layer

For *j*-th layer:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \cos\delta_j & \dfrac{i\sin\delta_j}{\gamma_j} \\ i\gamma_j\sin\delta_j & \cos\delta_j \end{pmatrix}$$

For the whole structure:

$$\begin{pmatrix} E_{01} \\ \eta_0 H_{01} \end{pmatrix} = M_1 M_2 \ldots M_N \begin{pmatrix} E_{N,N+1} \\ \eta_0 H_{N,N+1} \end{pmatrix} = M_{tot} \begin{pmatrix} E_{N,N+1} \\ \eta_0 H_{N,N+1} \end{pmatrix}$$

Reflection and transmission coefficients

$$\begin{pmatrix} E_{01} \\ \eta_0 H_{01} \end{pmatrix} = \begin{pmatrix} E_{i0} + E_{r0} \\ (E_{i0} - E_{r0})\gamma_0 \end{pmatrix} = M_{tot} \begin{pmatrix} E_t \\ \gamma_t E_t \end{pmatrix} = M_{tot} \begin{pmatrix} E_{N,N+1} \\ \eta_0 H_{N,N+1} \end{pmatrix}$$

$$r = \frac{\gamma_0 m_{11} + \gamma_0 \gamma_t m_{12} - m_{21} - \gamma_t m_{22}}{\gamma_0 m_{11} + \gamma_0 \gamma_t m_{12} + m_{21} + \gamma_t m_{22}}$$

$$t = \frac{2\gamma_0}{\gamma_0 m_{11} + \gamma_0 \gamma_t m_{12} + m_{21} + \gamma_t m_{22}}$$

polarisation TE:	$\gamma_j = n_j \cos \alpha_j$	$\delta_j = \omega n_j d_j \cos \alpha_j / c$
polarisation TM:	$\gamma_j = n_j / \cos \alpha_j$	$\delta_j = \omega n_j d_j \cos \alpha_j / c$
normal incidence:	$\gamma_j = n_j$	$\delta_j = \omega n_j d_j / c$

Generalization

The formalism is also valid for

- absorbing layers; j-th layer absorbs: $N_j = n_j - i\kappa_j$

$$\cos\alpha_j = \frac{\sqrt{N_j^2 - n_0^2 \sin^2\alpha_0}}{N_j}$$

- layers where total reflection occurs; total reflection on j-th layer:

$$\cos\alpha_j = -i\frac{\sqrt{n_0 \sin^2\alpha_0 - n_j^2}}{n_j}$$

δ_j and γ_j become imaginary; one introduces: $\Delta_j = i\delta_j$ and $\Gamma_j = i\gamma_j$, where Δ_j and Γ_j are real. The transfer matrix becomes:

$$M_j = \begin{pmatrix} \operatorname{ch}\Delta_j & i\dfrac{\operatorname{sh}\Delta_j}{\Gamma_j} \\ -i\Gamma_j \operatorname{sh}\Delta_j & \operatorname{ch}\Delta_j \end{pmatrix}$$

Applications

Optical elements and coatings are designed for given incidence angle

Specific layer thicknesses are frequently used in stacks:

- quarter-wave (λ/4) layer

$$\delta = \frac{2\pi}{\lambda} \underbrace{n_1 d_1 \cos\alpha_1}_{\lambda/4} = \frac{\pi}{2}$$

$$M = \begin{pmatrix} 0 & i/\gamma \\ i\gamma & 0 \end{pmatrix}$$

- half-wave (λ/2) layer

$$\delta = \frac{2\pi}{\lambda} \underbrace{n_1 d_1 \cos\alpha_1}_{\lambda/2} = \pi$$

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- quarter-wave (λ/4-λ/4) bilayer

$$M = \begin{pmatrix} 0 & i/\gamma_1 \\ i\gamma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i/\gamma_2 \\ i\gamma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\gamma_2/\gamma_1 & 0 \\ 0 & -\gamma_1/\gamma_2 \end{pmatrix}$$

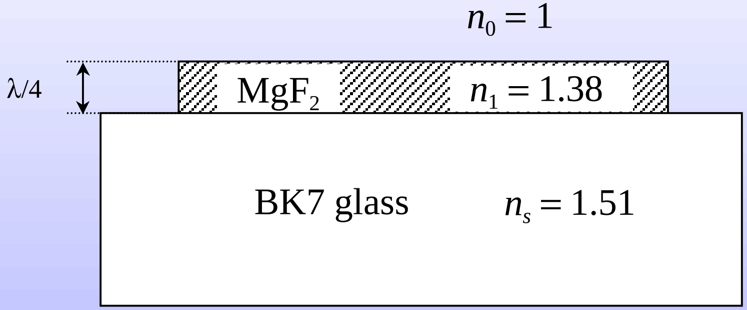
Antireflective single layer

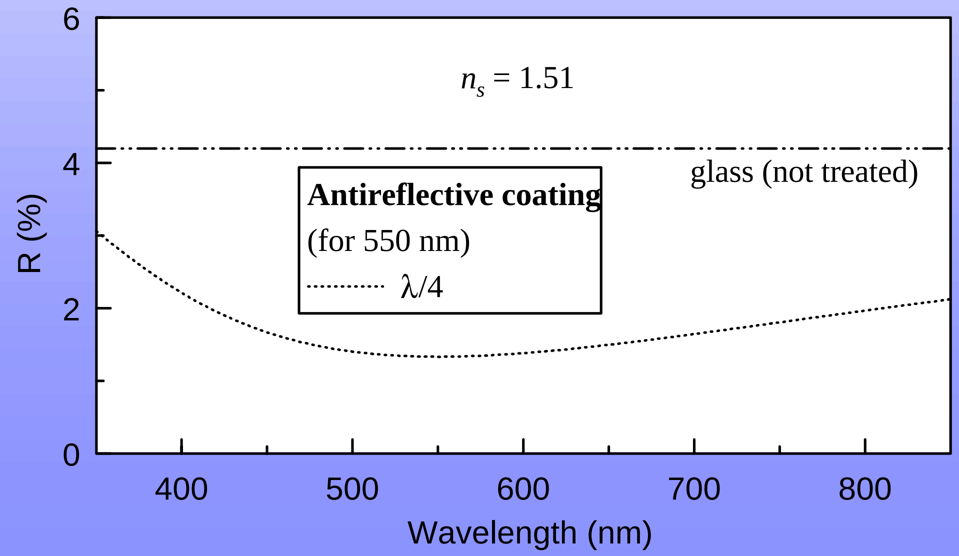
Let's try λ/4-layer (then waves with λ/2 phase delay will interfere)

$$M = \begin{pmatrix} 0 & i/\gamma_1 \\ i\gamma_1 & 0 \end{pmatrix}$$

$$r = \frac{\gamma_0\gamma_s - \gamma_1^2}{\gamma_0\gamma_s + \gamma_1^2} = \frac{n_s - n_1^2}{n_s + n_1^2}$$

$$n_1 = \sqrt{n_s}$$





Broadband

Less efficient (no degree of freedom for n_1)

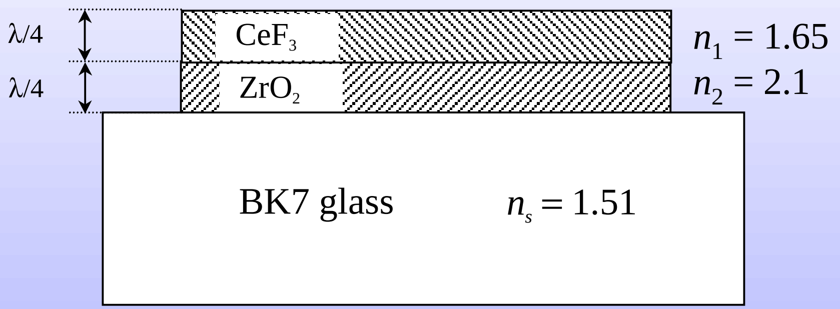
Antireflective bilayer

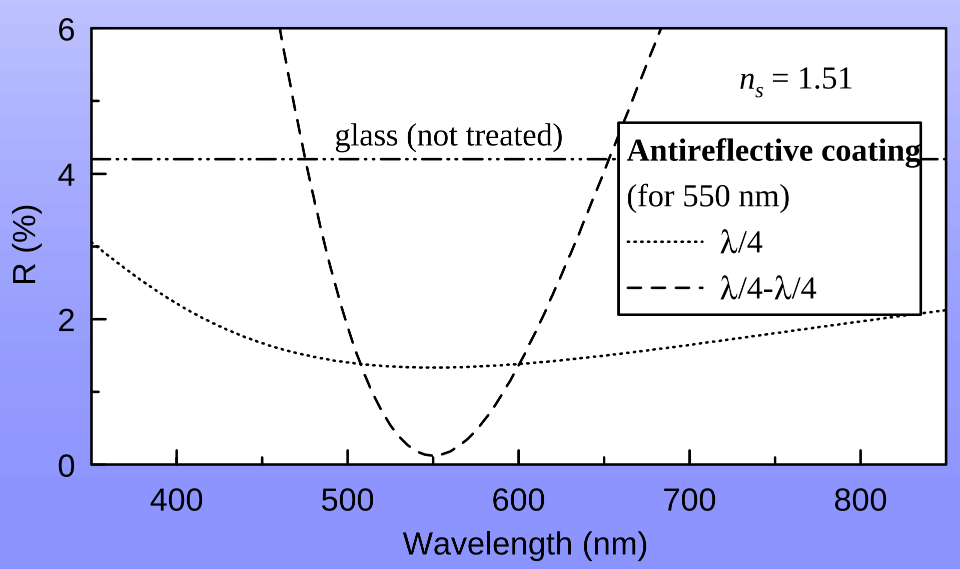
- quarter-wave (λ/4-λ/4) bilayer

$$M = \begin{pmatrix} -\gamma_2/\gamma_1 & 0 \\ 0 & -\gamma_1/\gamma_2 \end{pmatrix}$$

$$r = \frac{\gamma_2^2\gamma_0 - \gamma_s\gamma_1^2}{\gamma_2^2\gamma_0 + \gamma_s\gamma_1^2} = \frac{n_2^2 - n_s n_1^2}{n_2^2 + n_s n_1^2}$$

$$\frac{n_2}{n_1} = \sqrt{n_s}$$





V-like shape (narrow frequency range)

More efficient (one degree of freedom for n_1, n_2)

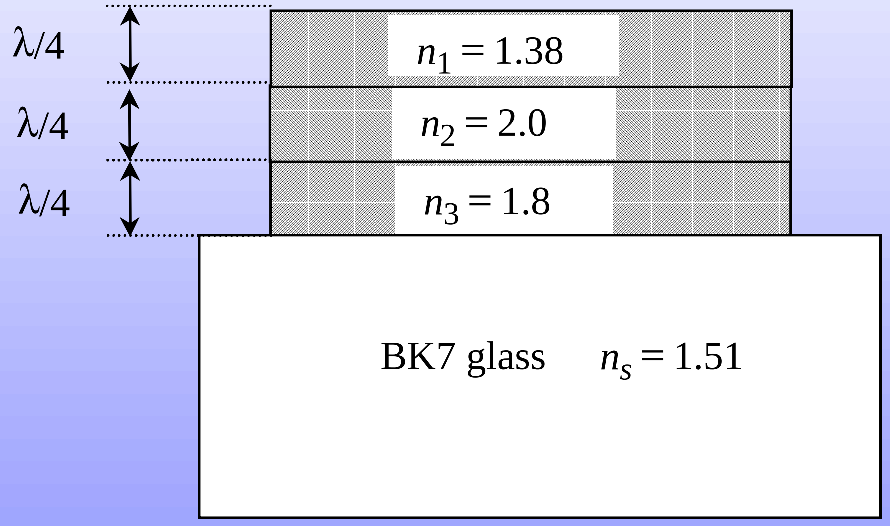
Broadband AR coating

- trilayer structure (λ/4-λ/4-λ/4)

$$M = \begin{pmatrix} 0 & -i\dfrac{\gamma_2}{\gamma_1\gamma_3} \\ -i\dfrac{\gamma_1\gamma_3}{\gamma_2} & 0 \end{pmatrix}$$

$$r = \frac{\gamma_2^2\gamma_0\gamma_s - \gamma_1^2\gamma_3^2}{\gamma_2^2\gamma_0\gamma_s + \gamma_1^2\gamma_3^2} = \frac{n_2^2 n_s - n_1^2 n_3^2}{n_2^2 n_s + n_1^2 n_3^2}$$

$$\frac{n_1 n_3}{n_2} = \sqrt{n_s}$$



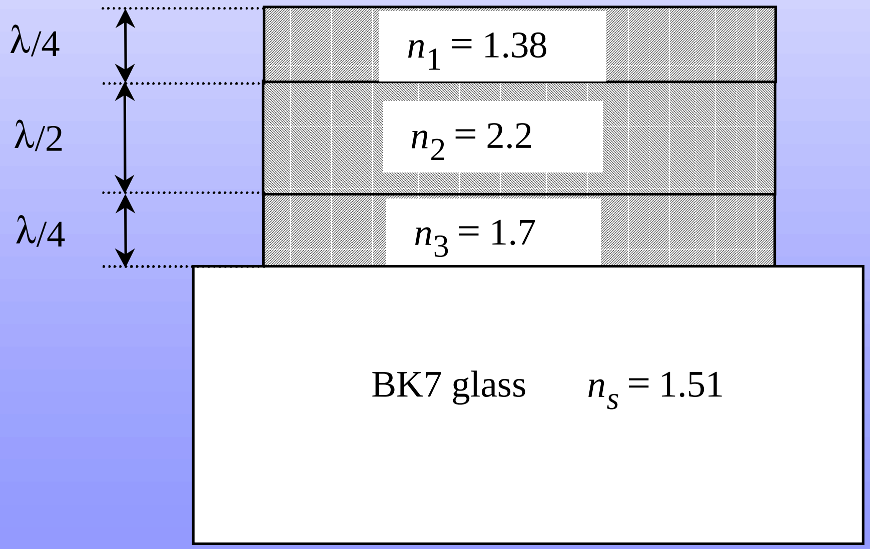
Broadband AR coating

- trilayer structure (λ/4-λ/2-λ/4): similar to a quarter-wave bilayer at the resonant wavelength
- half-wave layer helps to extend the antireflective range

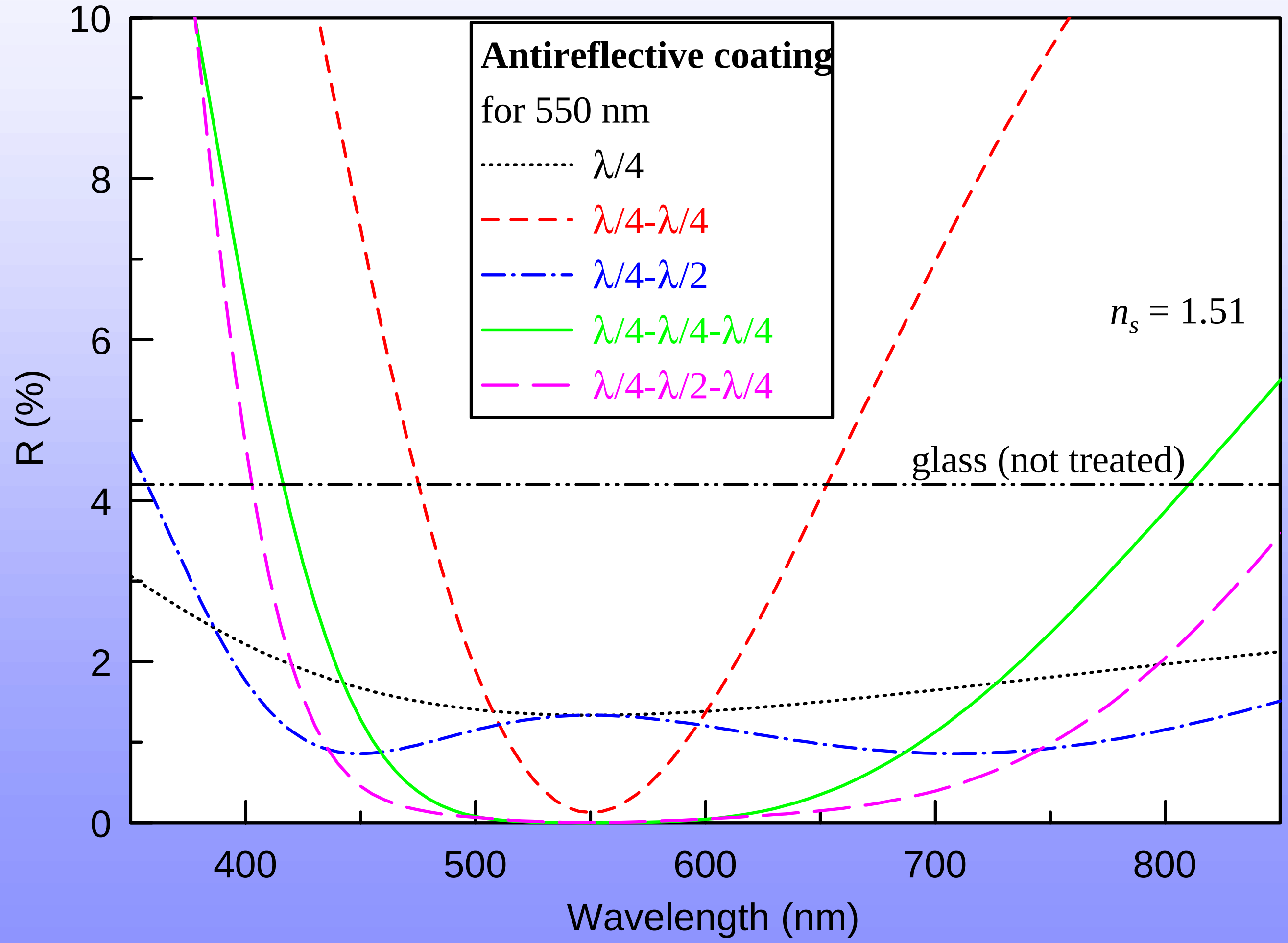
$$M = \begin{pmatrix} \gamma_3/\gamma_1 & 0 \\ 0 & \gamma_1/\gamma_3 \end{pmatrix}$$

$$r = \frac{-\gamma_3^2\gamma_0 + \gamma_s\gamma_1^2}{\gamma_3^2\gamma_0 + \gamma_s\gamma_1^2} = \frac{n_s n_1^2 - n_3^2}{n_3^2 + n_s n_1^2}$$

$$\frac{n_3}{n_1} = \sqrt{n_s}$$



Antireflective coating: summary



Dielectric mirrors

1 bilayer ($n_L << n_H$):

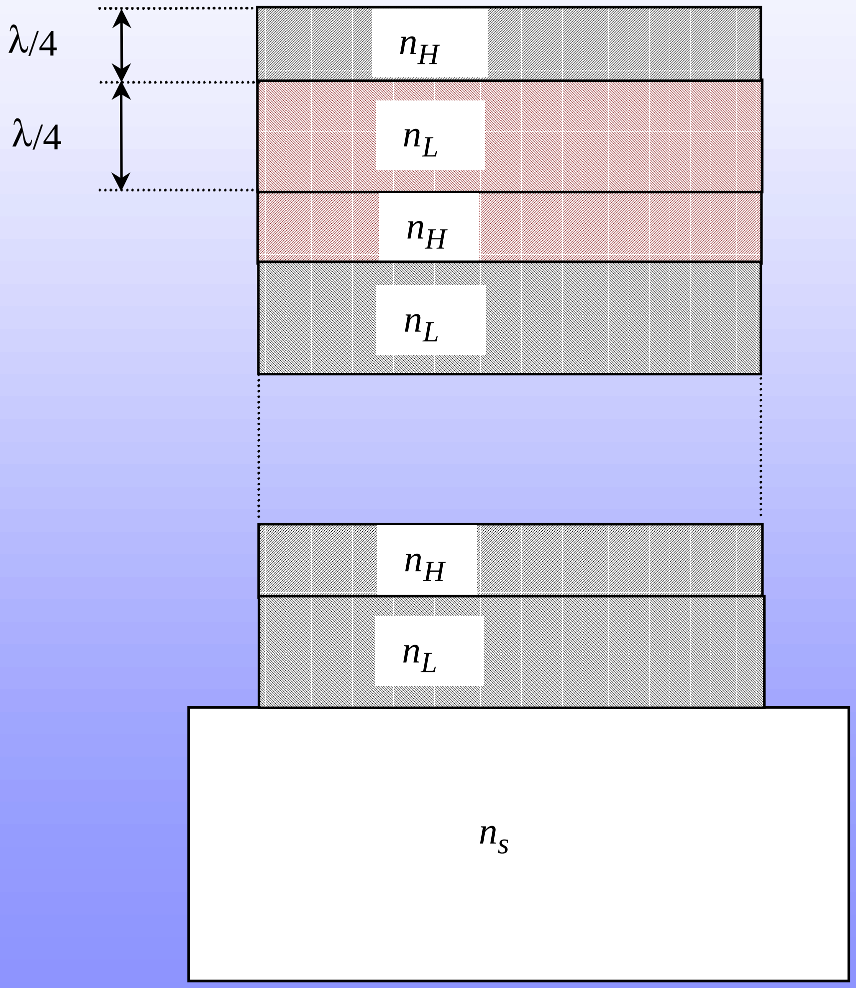
$$M = \begin{pmatrix} -n_L/n_H & 0 \\ 0 & -n_H/n_L \end{pmatrix}$$

N bilayers:

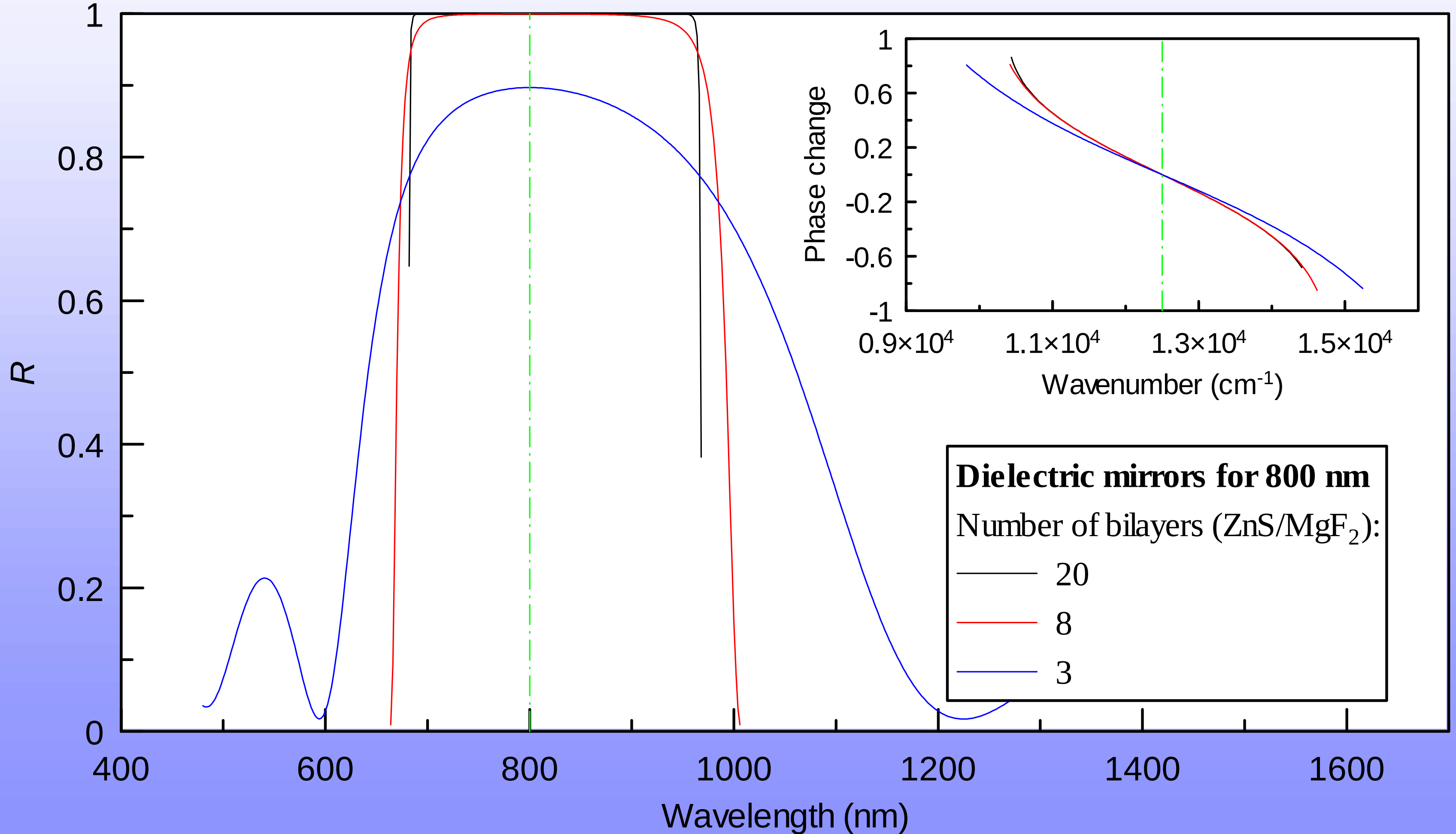
$$M = \begin{pmatrix} (-n_L/n_H)^N & 0 \\ 0 & (-n_H/n_L)^N \end{pmatrix}$$

Reflectivity:

$$R = \left(\frac{(1/n_s)(n_L/n_H)^{2N} - 1}{(1/n_s)(n_L/n_H)^{2N} + 1} \right)^2$$



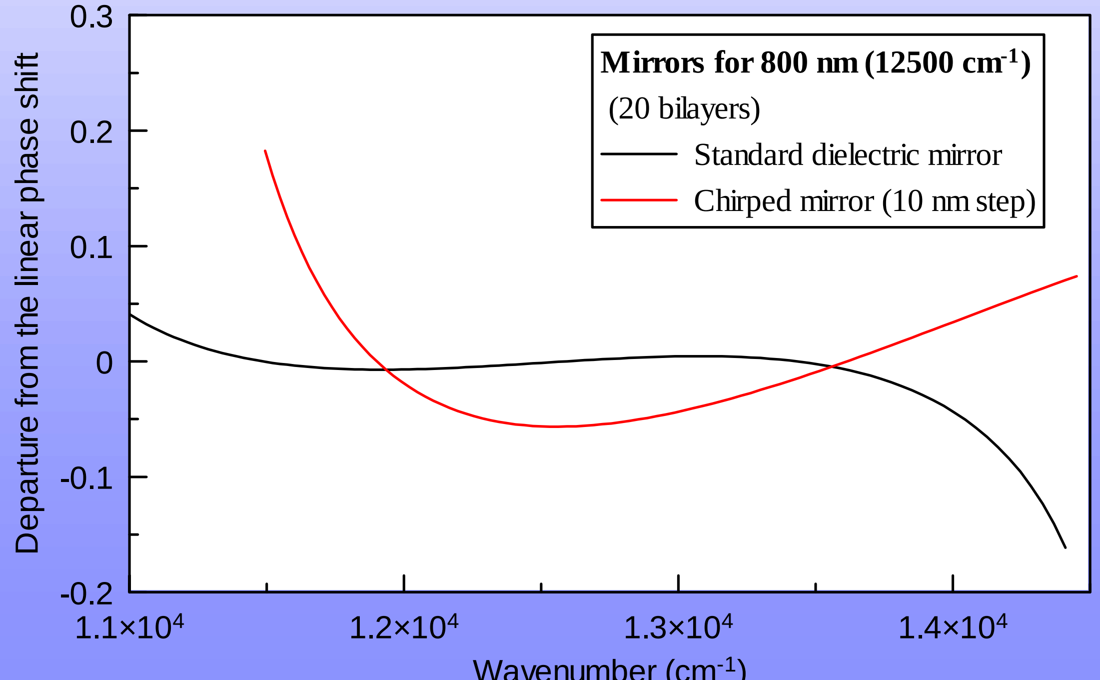
Dielectric mirrors: example



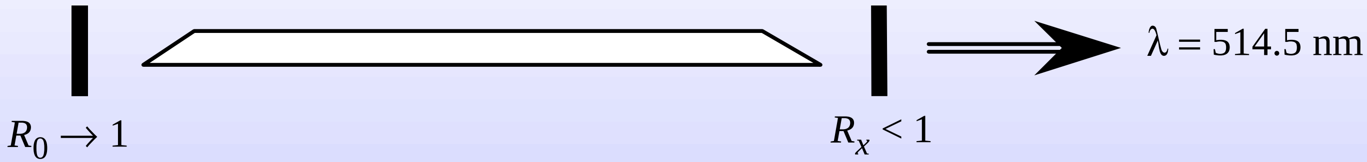
Chirped dielectric mirrors

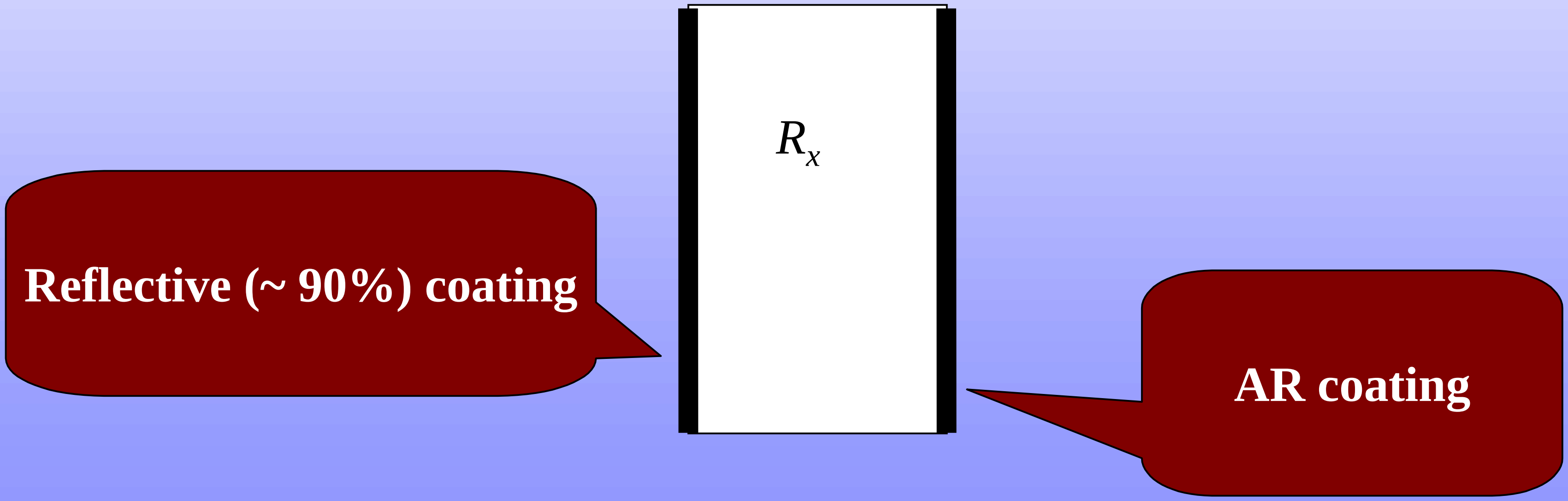
The resonant wavelength is linearly tuned along the stack of bilayers

- Different wavelengths are reflected at different depths → different optical paths
- Adding or compensating of a chirp of the pulses

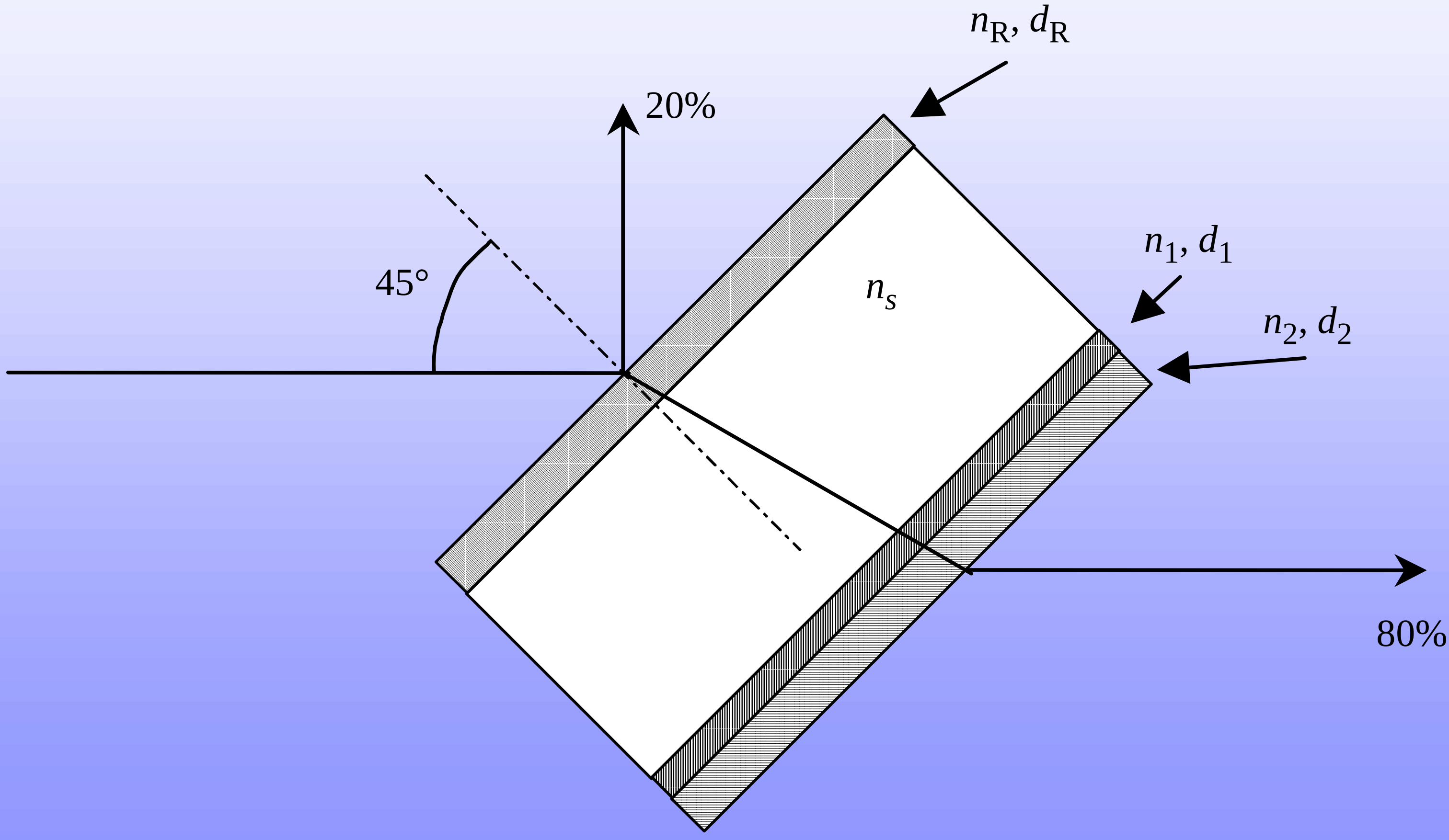


Laser output coupler





Beamsplitter



Beamsplitting mirror

- Separation of harmonic frequencies (e.g. Nd:YAG fundamental beam at 1064 nm and its second harmonic at 532 nm)
- Long-pass or cut-off filters

Example of solution:

1) Stack of $\lambda/4$ bilayers forming a dielectric mirror for 1064 nm
   These layers are $\lambda/2$ for the second harmonics so it passes through unchanged

2) We add below an AR coating for 532 nm
   1064 nm component does not penetrate down to these layers

Result: 1064 nm is reflected, 532 nm is transmitted

- Sometimes a detuning of a resonant wavelength is used in several layers of the HR coating: it can smooth the unwanted interference maxima and minima.

Interference band-pass filters

Contain:

- Stacks of high-reflecting bilayers
- Antireflective coatings
- Fabry-Pérot cavities
- Detuning of the resonant wavelength is also often used for smoothing of the interferences