Lecture 5: Interfaces between LHI media

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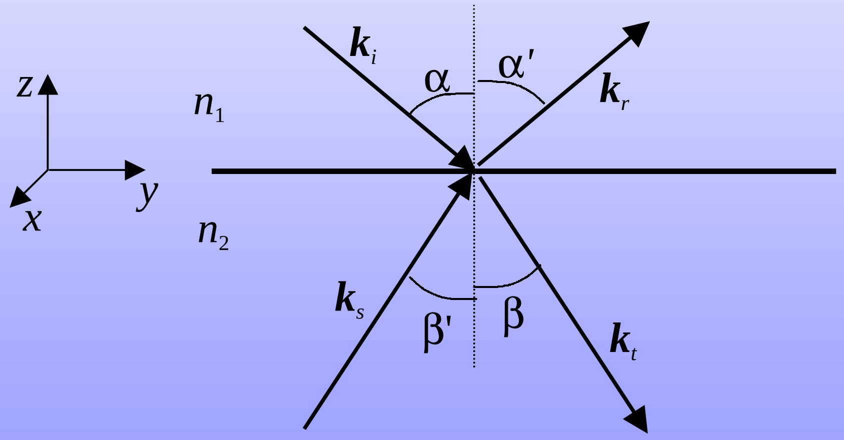
- Eigenmodes of electromagnetic field near interfaces: boundary conditions
- Snell's law, Fresnel formulae
- Total reflection, phase shifts
  - ☞ Application: Fresnel prism
- Absorbing media
  - ☞ Application: metal mirrors
- Two parallel interfaces: Fabry-Pérot etalon

Interface: eigenmodes

$$\sum \boldsymbol{E}_j \, e^{i(\omega_j t - \boldsymbol{k}_j \cdot \boldsymbol{r})} \qquad k_j = \frac{\omega_j}{c} n_1$$



$$\sum \boldsymbol{E}_j \, e^{i(\omega_j t - \boldsymbol{k}_j \cdot \boldsymbol{r})} \qquad k_j = \frac{\omega_j}{c} n_2$$

Boundary conditions:

$$E_{x1} = E_{x2}$$
$$E_{y1} = E_{y2}$$
$$D_{z1} = D_{z2}$$
$$H_{x1} = H_{x2}$$
$$H_{y1} = H_{y2}$$
$$B_{z1} = B_{z2}$$

For any t:

$\omega = const$

- For any x, y in the interface plane:

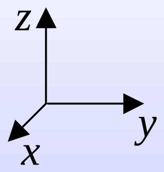
$k_x, k_y = const$

- Possible k_z:

$$k_z = \pm\sqrt{\frac{\omega^2}{c^2} n_{1,2}^2 - k_y^2 - k_x^2}$$

Snell's law

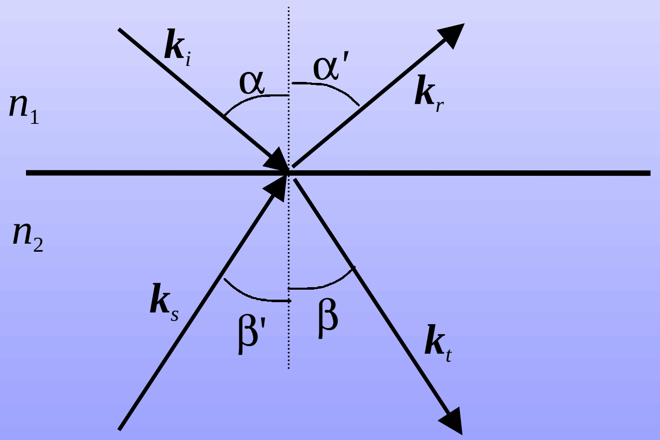
- Choice of the system of axes:



$$k_x = 0$$

- Condition $k_y = k_i \sin\alpha = const$:

$$\alpha = \alpha', \qquad n_1 \sin\alpha = n_2 \sin\beta, \qquad \beta = \beta'$$



- z-components of the wave vector:

$$k_{z,i} = -\sqrt{n_1^2\omega^2/c^2 - k_y^2} = -n_1\frac{\omega}{c}\cos\alpha$$

$$k_{z,r} = \sqrt{n_1^2\omega^2/c^2 - k_y^2} = n_1\frac{\omega}{c}\cos\alpha$$

$$k_{z,t} = -\sqrt{n_2^2\omega^2/c^2 - k_y^2} = -n_2\frac{\omega}{c}\cos\beta$$

$$k_{z,s} = \sqrt{n_2^2\omega^2/c^2 - k_y^2} = n_2\frac{\omega}{c}\cos\beta$$

- All k_z-components can be both real and imaginary

Wave vector and field components

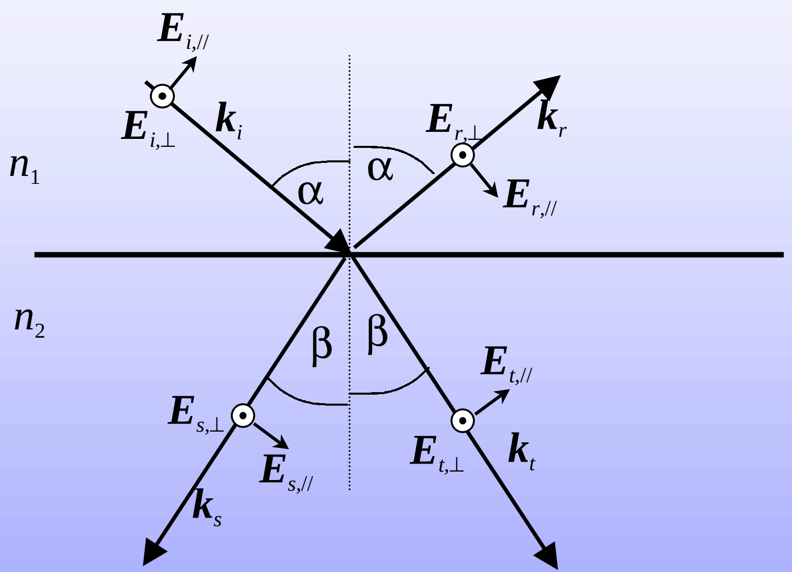
$$\boldsymbol{k}_i = \frac{\omega n_1}{c}\begin{pmatrix}0 & s_y^{(1)} & -s_z^{(1)}\end{pmatrix}$$

$$\boldsymbol{k}_r = \frac{\omega n_1}{c}\begin{pmatrix}0 & s_y^{(1)} & s_z^{(1)}\end{pmatrix}$$

$$\boldsymbol{k}_t = \frac{\omega n_2}{c}\begin{pmatrix}0 & s_y^{(2)} & -s_z^{(2)}\end{pmatrix}$$

$$\boldsymbol{k}_s = \frac{\omega n_2}{c}\begin{pmatrix}0 & s_y^{(2)} & s_z^{(2)}\end{pmatrix}$$

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad s_y^{(1)} = \sin\alpha, \quad s_z^{(1)} = \cos\alpha, \quad s_y^{(2)} = \sin\beta, \quad s_z^{(2)} = \cos\beta$$



$$\boldsymbol{E}_i = \begin{pmatrix}E_{\perp,i} & E_{//,i}\, s_z^{(1)} & E_{//,i} s_y^{(1)}\end{pmatrix}$$

$$\boldsymbol{E}_r = \begin{pmatrix}E_{\perp,r} & -E_{//,r}\, s_z^{(1)} & E_{//,r} s_y^{(1)}\end{pmatrix}$$

$$\boldsymbol{E}_t = \begin{pmatrix}E_{\perp,t} & E_{//,t}\, s_z^{(2)} & E_{//,i} s_y^{(2)}\end{pmatrix}$$

$$\boldsymbol{E}_s = \begin{pmatrix}E_{\perp,s} & -E_{//,s}\, s_z^{(2)} & E_{//,s} s_y^{(2)}\end{pmatrix}$$

$$\boldsymbol{H} = \frac{n}{\eta_0}\left(\boldsymbol{s} \wedge \boldsymbol{E}\right)$$

Continuity conditions

$$E_{x1} = E_{x2} \qquad E_{\perp,i} + E_{\perp,r} = E_{\perp,t} + E_{\perp,s}$$

$$H_{y1} = H_{y2} \qquad n_1 s_z^{(1)}\left(E_{\perp,i} - E_{\perp,r}\right) = n_2 s_z^{(2)}\left(E_{\perp,t} - E_{\perp,s}\right)$$

$$B_{z1} = B_{z2} \qquad n_1 s_y^{(1)}\left(E_{\perp,i} + E_{\perp,r}\right) = n_2 s_y^{(2)}\left(E_{\perp,t} + E_{\perp,s}\right)$$

2 independent equations

$$E_{y1} = E_{y2} \qquad s_z^{(1)}\left(E_{//,i} - E_{//,r}\right) = s_z^{(2)}\left(E_{//,t} - E_{//,s}\right)$$

$$D_{z1} = D_{z2} \qquad n_1^2 s_y^{(1)}\left(E_{//,i} + E_{//,r}\right) = n_2^2 s_y^{(2)}\left(E_{//,t} + E_{//,s}\right)$$

$$H_{x1} = H_{x2} \qquad n_1\left(E_{//,i} + E_{//,r}\right) = n_2\left(E_{//,t} + E_{//,s}\right)$$

2 independent equations

2 independent polarizations: perpendicular (⊥, TE, *s*) and parallel (//, TM, *p*)

- All wave vectors are real: 4 coupled waves (2 independent + 2 dependent); usual choice: $E_s = 0$, E_i = incident, E_t = transmitted, E_r = reflected.
- Two wave vectors are imaginary: 3 coupled waves (1 independent + 2 dependent); E_s rejected (divergent), E_i = incident, E_t = evanescent, E_r = totally reflected

Fresnel equations

$$r_\perp = \frac{n_1\cos\alpha - n_2\cos\beta}{n_1\cos\alpha + n_2\cos\beta}$$

$$r_\perp \equiv r_s = -\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$$

$$t_\perp = \frac{2n_1\cos\alpha}{n_1\cos\alpha + n_2\cos\beta}$$

$$t_\perp \equiv t_s = \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$$r_{//} = \frac{n_1\cos\beta - n_2\cos\alpha}{n_1\cos\beta + n_2\cos\alpha}$$

$$r_{//} \equiv r_p = -\frac{\mathrm{tg}(\alpha-\beta)}{\mathrm{tg}(\alpha+\beta)}$$

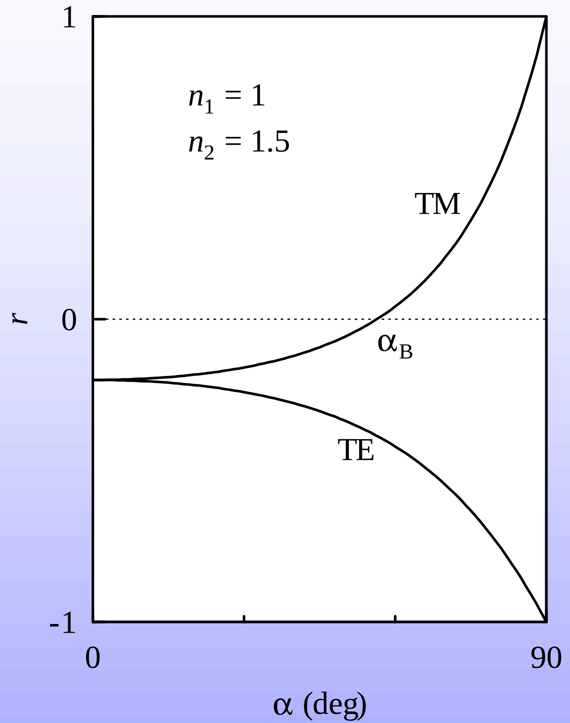
$$t_{//} = \frac{2n_1\cos\alpha}{n_1\cos\beta + n_2\cos\alpha}$$

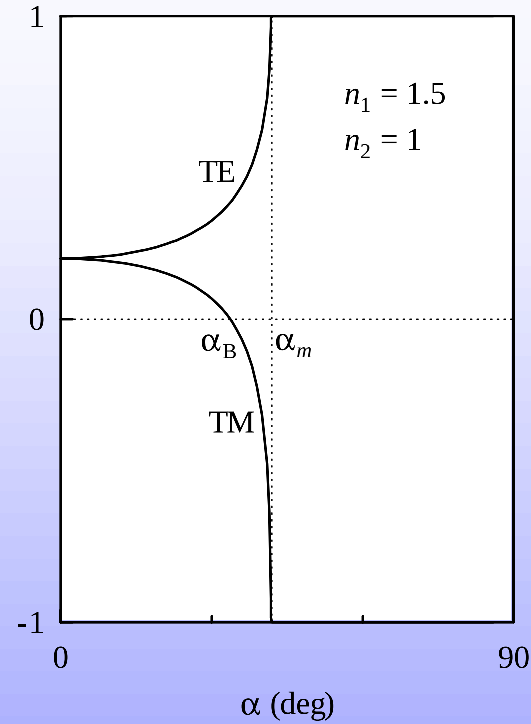
$$t_{//} \equiv t_p = \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)\cos(\alpha-\beta)}$$

$$R_{\perp,//} = \frac{S_r\cos\alpha}{S_i\cos\alpha} = \frac{E_r^2}{E_i^2} = r_{\perp,//}^2$$

$$T_{\perp,//} = \frac{S_t\cos\beta}{S_i\cos\alpha} = \frac{n_2\cos\beta\,E_t^2}{n_1\cos\alpha\,E_i^2} = \frac{n_2\cos\beta}{n_1\cos\alpha}t_{\perp,//}^2$$

Reflected waves





Critical angle α_m :

$$\sin\alpha_m = n_2/n_1$$

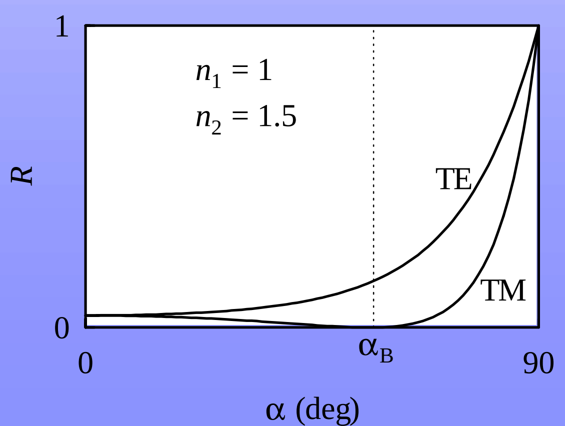
for $\alpha > \alpha_m$ the wave is totally reflected

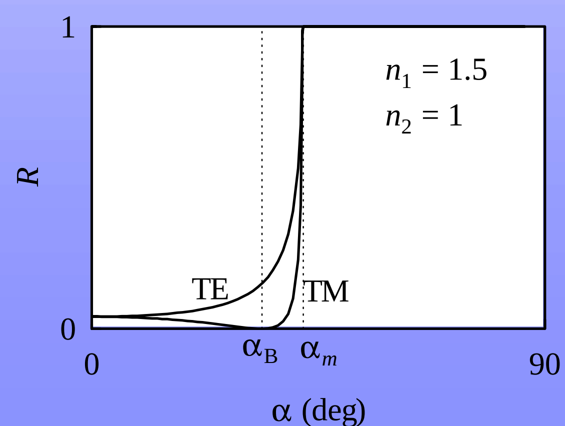
Brewster angle α_B :

$$tg\alpha_B = n_2/n_1$$

$$\alpha_B + \beta_B = \pi/2$$

$$r_{//} = 0$$





Problem of the phase shifts after reflection:

if $\boldsymbol{E}$ is out of phase (shifted by π) then $\boldsymbol{B}$ is in phase and vice versa.

Total reflection

$$k_{z,t}\left(=-n_2\frac{\omega}{c}\cos\beta\right)=-\sqrt{\frac{\omega^2}{c^2}n_2^2-k_y^2}=-n_1\frac{\omega}{c}\sqrt{n^2-\sin^2\alpha}=i\,n_1\frac{\omega}{c}\sqrt{\sin^2\alpha-n^2}$$

with $n = n_2/n_1 < 1$

$$\boldsymbol{E}_t=\boldsymbol{E}_{0t}\,e^{i(\omega t-\boldsymbol{k}_t\cdot\boldsymbol{r})}=\boldsymbol{E}_{0t}\,e^{i\left(\omega t-\boldsymbol{k}_{t,y}y\right)}\,e^{-i\boldsymbol{k}_{t,z}z}=\boldsymbol{E}_{0t}\,e^{i\left(\omega t-\boldsymbol{k}_{t,y}y\right)}\,e^{\frac{z}{2h}}$$

$$\langle S\rangle \propto e^{-\frac{|z|}{h}}$$

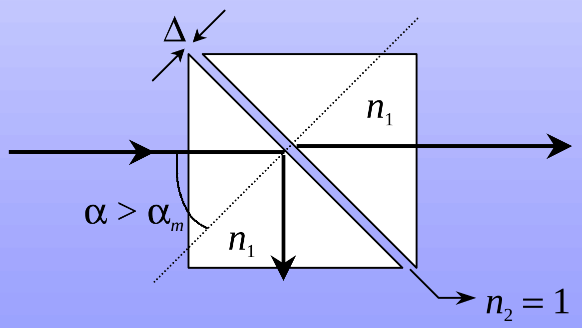
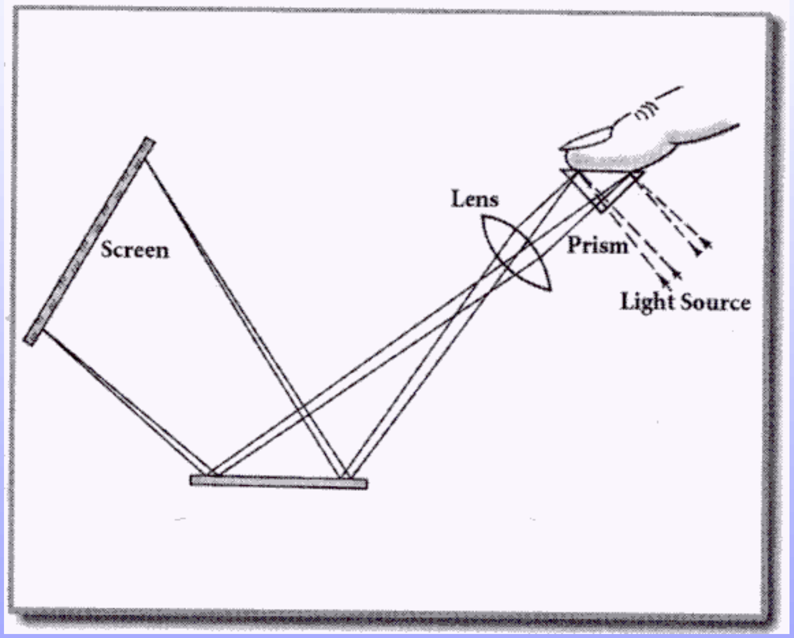


Illustration...

$$\cos\beta=-i\frac{\sqrt{\sin^2\alpha-n^2}}{n}$$

$\cos\beta$ is then replaced in Fresnel formulae by this new definition

Total internal reflection: illustration



Total reflection: continued

$$r_{\perp} = \frac{n_1 \cos\alpha - n_2 \cos\beta}{n_1 \cos\alpha + n_2 \cos\beta} \rightarrow \frac{\cos\alpha + i\sqrt{\sin^2\alpha - n^2}}{\cos\alpha - i\sqrt{\sin^2\alpha - n^2}}$$

$$r_{//} = \frac{n_1 \cos\beta - n_2 \cos\alpha}{n_1 \cos\beta + n_2 \cos\alpha} \rightarrow -\frac{n^2\cos\alpha + i\sqrt{\sin^2\alpha - n^2}}{n^2\cos\alpha - i\sqrt{\sin^2\alpha - n^2}}$$

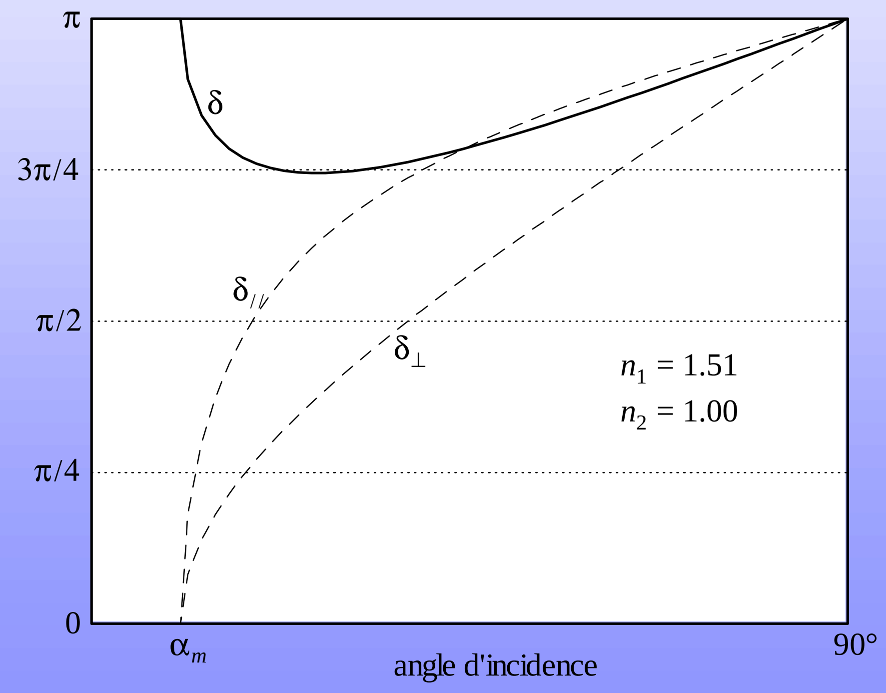
$$r_j = \pm\frac{|r_j|\, e^{i\delta_j/2}}{|r_j|\, e^{-i\delta_j/2}} = \pm e^{i\delta_j}$$

Dephasing between ⊥ and // components: $\delta = \delta_{\perp} - \delta_{//} + \pi$

$$\mathrm{tg}\left(\frac{\delta_{\perp}}{2}\right) = \frac{\sqrt{\sin^2\alpha - n^2}}{\cos\alpha} \qquad \mathrm{tg}\left(\frac{\delta_{//}}{2}\right) = \frac{\sqrt{\sin^2\alpha - n^2}}{n^2\cos\alpha}$$

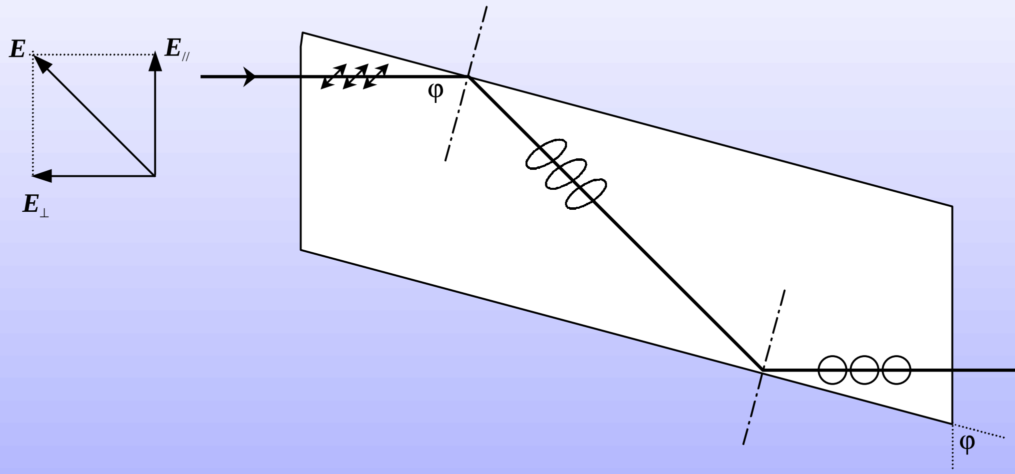
Total reflection: dephasing

$$\mathrm{tg}\left(\frac{\delta}{2}-\frac{\pi}{2}\right)=\mathrm{tg}\left(\frac{\delta_{\perp}}{2}-\frac{\delta_{//}}{2}\right)=-\frac{\cos\alpha\sqrt{\sin^2\alpha-n^2}}{\sin^2\alpha}$$

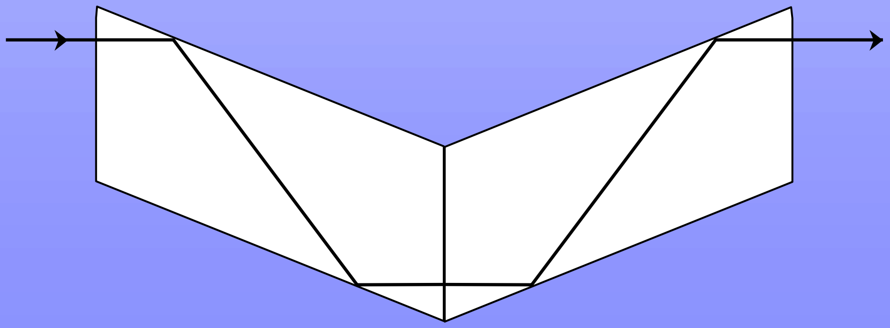


Total reflection: application

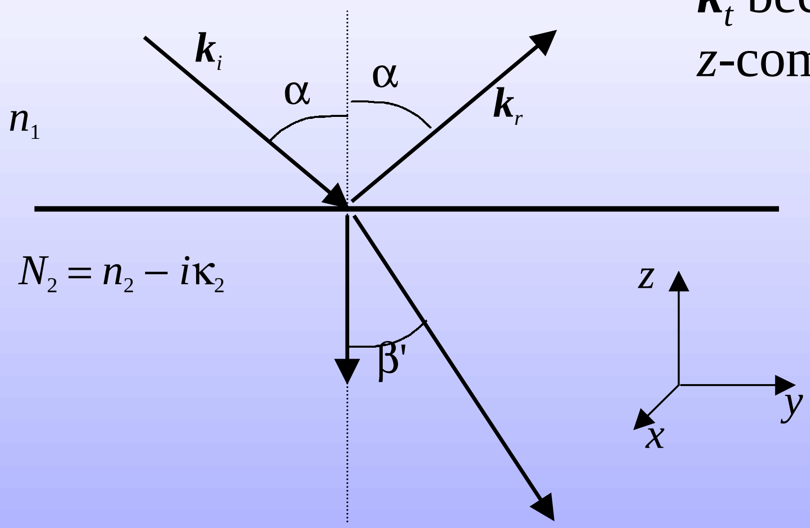
Fresnel prism: quasi-wavelength independent quarter-wave plate



Combination of 2 Fresnel prisms: half-wave plate



Reflection on an absorbing medium



$\boldsymbol{k}_t$ becomes complex (or strictly speaking its *z*-component becomes complex):

$$\boldsymbol{k}_t = \boldsymbol{k}_t' - i\boldsymbol{k}_t''$$

$$\boldsymbol{k}_t = \frac{\omega}{c}\left(0, \quad n_1 \sin\alpha, \quad -\sqrt{N_2^2 - n_1^2 \sin^2\alpha}\right)$$

We can formally introduce a complex angle β:

$$N_2 \cos\beta = \sqrt{N_2^2 - n_1^2 \sin^2\alpha}$$

$$N_2 \sin\beta = n_1 \sin\alpha$$

and use the previously derived Fresnel formulae

Reflection on an absorbing medium: continued

Two differences with non-absorbing medium should be pointed out:

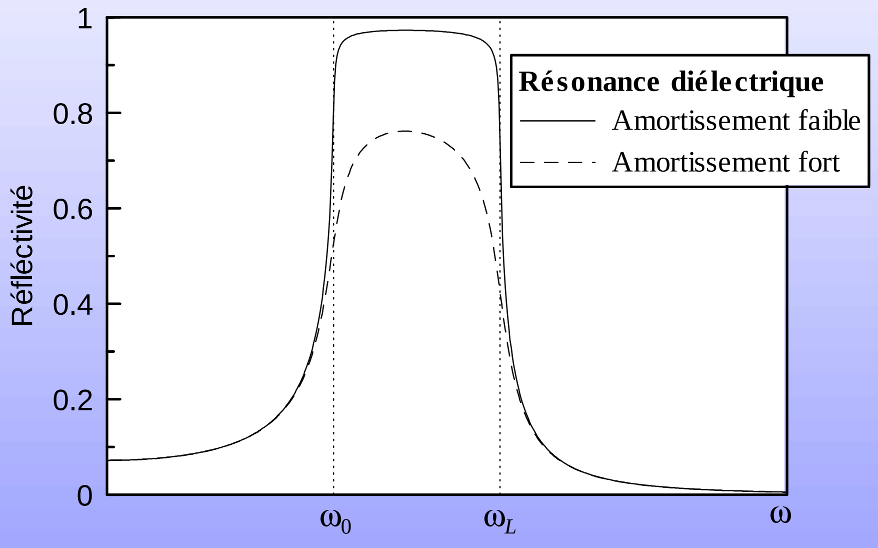
- There is a phase shift due to the absorption: $r_{\perp}$ and $r_{//}$ are complex for any angle of incidence
- The absorption increases the reflectivity. One gets for the normal incidence:

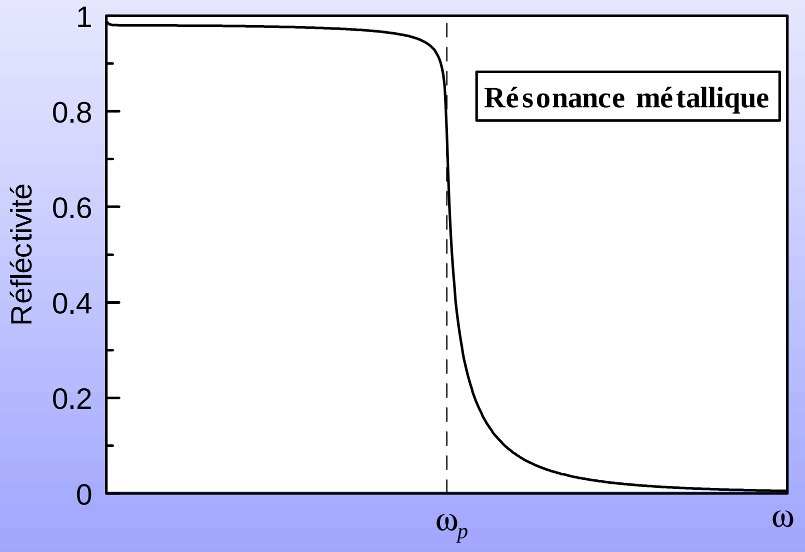
$$R = \left|\frac{N-1}{N+1}\right|^2 = \left(\frac{n-i\kappa-1}{n-i\kappa+1}\right)\left(\frac{n+i\kappa-1}{n+i\kappa+1}\right) = \frac{(n-1)^2+\kappa^2}{(n+1)^2+\kappa^2} > \frac{(n-1)^2}{(n+1)^2}$$

The reflectivity is very large ($R \rightarrow 1$) when:

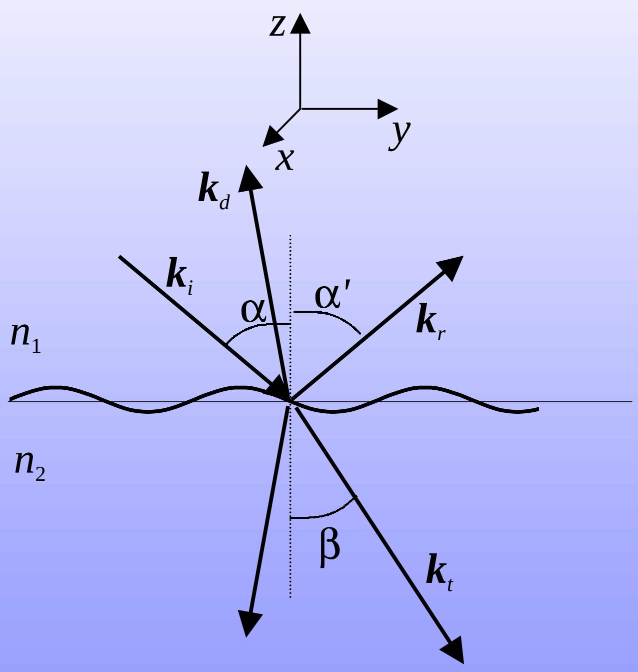
- The absorption index is large ($\kappa >> 1$): strongly absorbing medium does not absorb much — it reflects.
- The refractive index vanishes ($n \rightarrow 0$): this happens below the plasma frequency in metals and between the transverse and longitudinal resonance in dielectrics.

Reflectivity near resonances





Ondulating surface: second order effects



$$u(y,t) = u_0\, e^{i(\Omega t - Q y)}$$

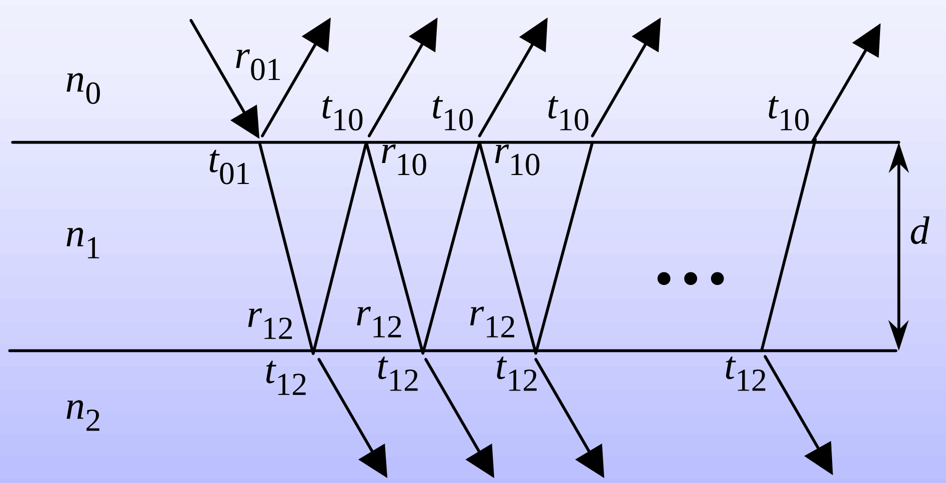
- Interface conditions: continuity at $z = u$
- First order development:

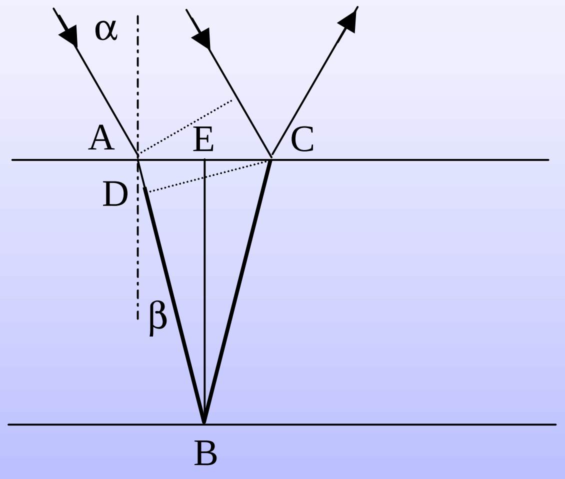
$$E(z=u) = E_0\, e^{i(\omega t - k_y y - k_z u)} \approx E_0 (1 - i k_z u)\, e^{i(\omega t - k_y y)} =$$
$$= E_0 \left(1 - i k_z u_0 e^{i(\Omega t - Q y)}\right) e^{i(\omega t - k_y y)}$$

- Waves with new wave vectors and frequencies appear:

$$k_{d,y} = k_y + Q$$
$$\omega_d = \omega + \Omega$$

Two interfaces: intuitive method





Difference of the beam paths:

$\Delta L\ n_1$

$\Delta L = \mathrm{ABC} - \mathrm{AD}$

$\mathrm{ABC} = 2d\ /\ \cos\beta$

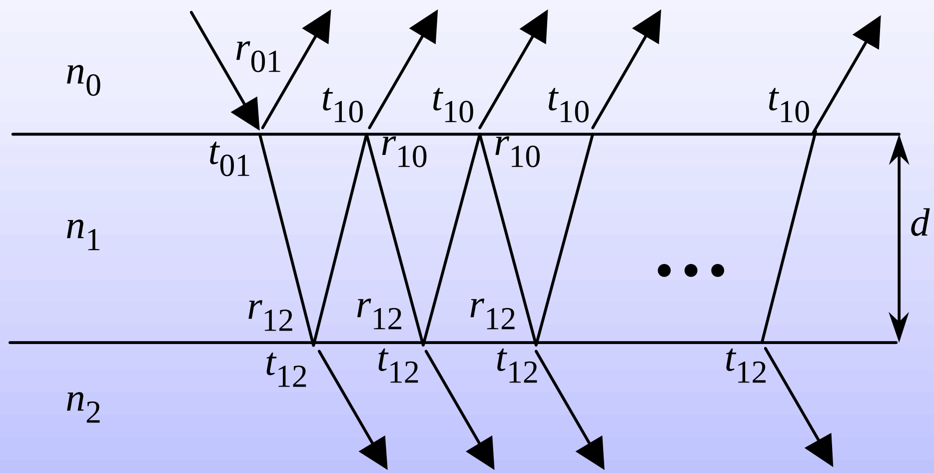
$\mathrm{AD} = \mathrm{AC}\ \sin\beta = 2\ \mathrm{AE}\ \sin\beta =$

$= 2d\ \sin\beta\ \mathrm{tg}\beta$

$\Delta L\ n_1 = 2dn_1\cos\beta$

Phase change: $e^{-} = e^{-2i\omega nd\cos\beta/c}$

Sum of partial waves



$$r = r_{01} + t_{01}t_{10}r_{12}e^{-} + t_{01}t_{10}r_{12}e^{-}\, r_{10}r_{12}e^{-} + t_{01}t_{10}r_{12}e^{-}\left(r_{10}r_{12}e^{-}\right)^2 \ldots =$$

$$= -r_{10} + t_{01}t_{10}r_{12}e^{-}\sum_{k=0}^{\infty}\left(e^{-}\, r_{10}r_{12}\right)^k = \frac{r_{12}e^{-2i\omega n_1 d\cos\beta/c} - r_{10}}{1 - r_{10}r_{12}e^{-2i\omega n_1 d\cos\beta/c}}$$

$$t = t_{01}t_{12} + t_{01}t_{12}r_{12}r_{10}e^{-} + t_{01}t_{12}\left(r_{12}r_{10}e^{-}\right)^2 + t_{01}t_{12}\left(r_{12}r_{10}e^{-}\right)^3 + \ldots =$$

$$= t_{01}t_{12}\sum_{k=0}^{\infty}\left(e^{-}\, r_{10}r_{12}\right)^k = \frac{t_{01}t_{12}}{1 - r_{10}r_{12}e^{-2i\omega n_1 d\cos\beta/c}}$$

Fabry-Pérot etalon

Plane-parallel plate in the air

$$t = t_{01}t_{12} + t_{01}t_{12}r_{12}r_{10}e^{-} + t_{01}t_{12}\left(r_{12}r_{10}e^{-}\right)^2 + t_{01}t_{12}\left(r_{12}r_{10}e^{-}\right)^3 + \ldots =$$

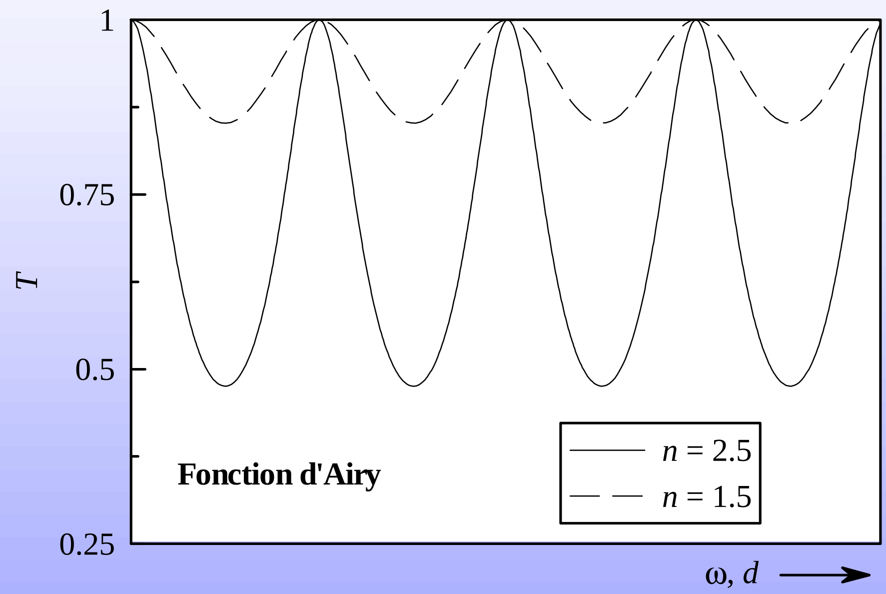
$$= t_{01}t_{12}\sum_{k=0}^{\infty}\left(e^{-}r_{10}r_{12}\right)^k = \frac{t_{01}t_{12}}{1 - r_{10}r_{12}e^{-2i\omega n_1 d\cos\beta/c}}$$

- Coefficients $T = tt^*$ et $R = rr^*$ are calculated
- $n_0 = n_2 = 1$
- $r_{10} = r_{12}$, $R_{10} = r_{10}$, $t_{10} = t_{01}$, $T_{01} = t_{01}$, $T_{10} = t_{10}$

Airy function:

$$T = \frac{T_{01}T_{10}}{\left(1 - R_{10}\right)^2 + 4R_{10}\sin^2\varphi} = \frac{1}{1 + \frac{4R_{10}}{\left(1 - R_{10}\right)^2}\sin^2\varphi} \qquad \left(\varphi = \omega n_1 d\cos\beta/c\right)$$

Airy function



$$T = \frac{1}{1 + \frac{4R_{10}}{(1 - R_{10})^2} \sin^2 \varphi}$$

Maximum ($T_{max} = 1$):

$$\varphi = m\,\pi$$

$$2n_1 d \cos\beta = m\,\lambda$$

$$\Delta l + \lambda/2 = (2m + 1)\,\lambda/2$$

Minimum:

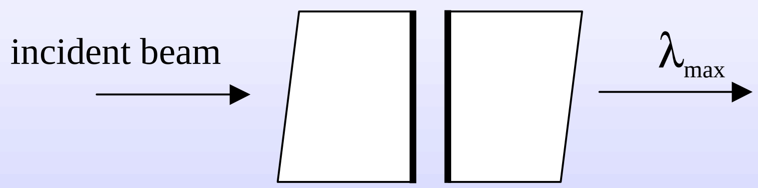
$$T_{\min} = \left(\frac{1 - R_{10}}{1 + R_{10}}\right)^2$$

$$\varphi = (2m - 1)\,\pi/2$$

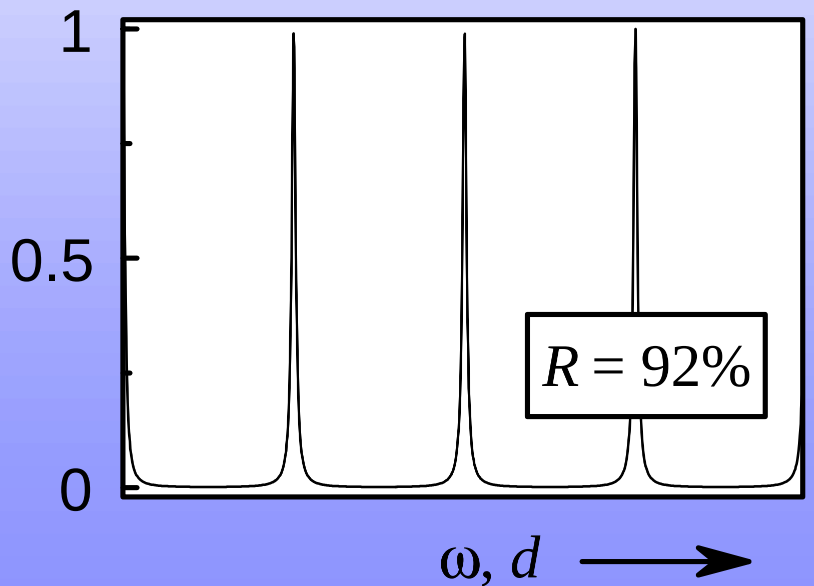
$$2n_1 d \cos\beta = (2m - 1)\,\lambda/2$$

$$\Delta l + \lambda/2 = m\lambda$$

Fabry-Pérot interferometer



Fabry-Pérot interferometer



$$\lambda_{max} = \frac{2nd}{m}$$

$$\text{Contrast}: \frac{T_{max}}{T_{min}} = \left(\frac{1+R_{10}}{1-R_{10}}\right)^2 \approx 550$$

Finesse: 37