Lecture 2: Propagation of light waves and pulses

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- Propagation of plane waves in linear homogeneous isotropic (LHI) media:
  - ☞ refractive index
  - ☞ intensity
  - ☞ absorption
- Propagation of light pulses:
  - ☞ group velocity
  - ☞ group dispersion — chirp
  - ☞ superluminal effects (?)

Wave equation

$$\left.\begin{aligned}\nabla\wedge\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0\\ \nabla\wedge\boldsymbol{H}-\frac{\partial\boldsymbol{D}}{\partial t}=0\end{aligned}\right\}\Rightarrow\quad\begin{aligned}\nabla^2\boldsymbol{E}-\varepsilon\mu_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}=0\\ \nabla^2\boldsymbol{H}-\varepsilon\mu_0\frac{\partial^2\boldsymbol{H}}{\partial t^2}=0\end{aligned}\quad\text{or}\quad\begin{aligned}\nabla^2\boldsymbol{E}-\frac{N^2}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}=0\\ \nabla^2\boldsymbol{H}-\frac{N^2}{c^2}\frac{\partial^2\boldsymbol{H}}{\partial t^2}=0\end{aligned}$$

$$\left(\nabla\wedge(\nabla\wedge\boldsymbol{E})=\nabla\nabla\cdot\boldsymbol{E}-\nabla^2\boldsymbol{E}\right)$$

well-known identity

$$\varepsilon=\varepsilon_0\varepsilon_r$$

$$\varepsilon_r=\varepsilon_r'-i\varepsilon_r''$$

dielectric constant

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$N^2=\varepsilon\mu_0c^2=\varepsilon_r$$

refractive index

$$N=n-i\kappa$$

$$\varepsilon_r'=n^2-\kappa^2,\qquad \varepsilon_r''=2n\kappa,$$

$$n=\sqrt{\frac{1}{2}\left(\sqrt{\varepsilon_r'^2+\varepsilon_r''^2}+\varepsilon_r'\right)},\qquad \kappa=\sqrt{\frac{1}{2}\left(\sqrt{\varepsilon_r'^2+\varepsilon_r''^2}-\varepsilon_r'\right)}$$

Propagation in the vacuum

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$
$$\nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$
$$\boldsymbol{H} = \boldsymbol{H}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

with $|\boldsymbol{k}| \equiv k = \frac{\omega}{c}$

- Back to Maxwell's equations:

$$\nabla \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = -i\boldsymbol{k} \wedge \boldsymbol{E} + i\omega\mu_0 \boldsymbol{H} = 0$$
$$\nabla \cdot \boldsymbol{E} = -i\boldsymbol{k} \cdot \boldsymbol{E} = 0$$

$$\frac{1}{\mu_0\omega}(\boldsymbol{k} \wedge \boldsymbol{E}_0) = \boldsymbol{H}_0$$

$$\boldsymbol{k} \perp \boldsymbol{E}_0 \perp \boldsymbol{H}_0 \perp \boldsymbol{k}$$
$$\boldsymbol{H}_0 = \eta_0^{-1}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$
$$\boldsymbol{B}_0 = c^{-1}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$

$$\boldsymbol{s} = \boldsymbol{k}/k$$
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$$

Superpositions of plane waves

- In the k-space: 3-dimensional

$$\boldsymbol{E}(t,\boldsymbol{r}) = \iiint \boldsymbol{E}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} d\boldsymbol{k}$$

$$\boldsymbol{H}(t,\boldsymbol{r}) = \iiint \boldsymbol{H}_0(\boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} d\boldsymbol{k}$$

- In the ω-space: 1-dimensional (propagation along z: $k_x = k_y = 0$)

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(\omega)\, e^{i(\omega t - kz)} d\omega$$

$$\boldsymbol{H}(t,z) = \int \boldsymbol{H}_0(\omega)\, e^{i(\omega t - kz)} d\omega$$

- Pulse: central frequency ω_0; central wave vector $k_0 = \omega_0/c$.

$$\boldsymbol{E}(t,z) = \boldsymbol{E}_1(t,z)\; e^{i(\omega_0 t - \boldsymbol{k}_0 z)} \quad = \quad e^{i(\omega_0 t - \boldsymbol{k}_0 z)} \int \boldsymbol{E}_0(\omega)\, e^{i((\omega-\omega_0)t - (k-k_0)z)} d\omega =$$

$$= e^{i(\omega_0 t - \boldsymbol{k}_0 z)} \int \boldsymbol{E}_0(\omega)\, e^{i(\omega-\omega_0)(t - z/c)} d\omega \quad = \quad \boldsymbol{E}_1(t - z/c)\; e^{i(\omega_0 t - \boldsymbol{k}_0 z)}$$

Propagation in the matter

$$\nabla^2 \boldsymbol{E} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$
$$\nabla^2 \boldsymbol{H} - \frac{N^2}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$
$$\boldsymbol{H} = \boldsymbol{H}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

with $|\boldsymbol{k}| \equiv k = \frac{\omega}{c} N$

- Back to Maxwell's equations:

$$\nabla \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = -i\boldsymbol{k} \wedge \boldsymbol{E} + i\omega\mu_0 \boldsymbol{H} = 0$$
$$\nabla \cdot \boldsymbol{E} = -i\boldsymbol{k} \cdot \boldsymbol{E} = 0$$

$$\frac{1}{\mu_0\omega}(\boldsymbol{k} \wedge \boldsymbol{E}_0) = \boldsymbol{H}_0$$

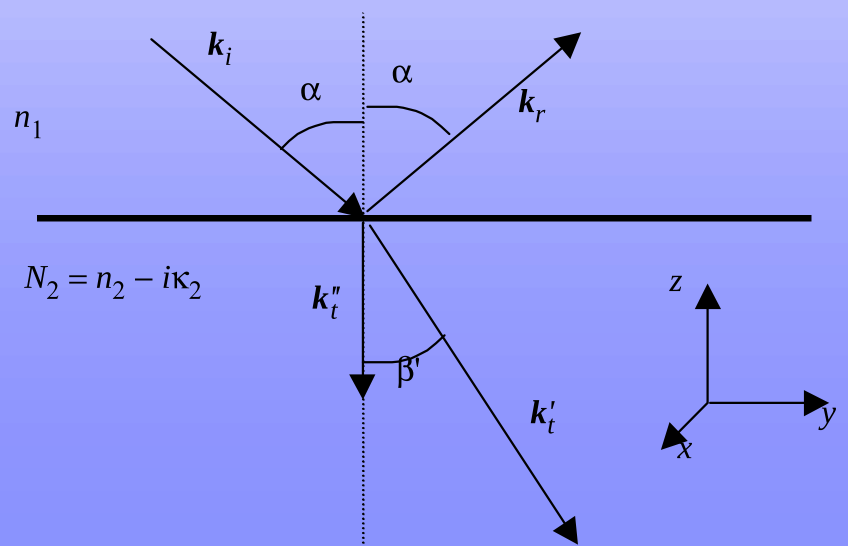
$$\boldsymbol{k} \perp \boldsymbol{E}_0 \perp \boldsymbol{H}_0 \perp \boldsymbol{k}$$
$$\boldsymbol{H}_0 = \frac{N}{\eta_0}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$
$$\boldsymbol{B}_0 = \frac{N}{c}(\boldsymbol{s} \wedge \boldsymbol{E}_0)$$

$$\boldsymbol{s} = \boldsymbol{k}/k$$
$$\eta = \frac{\eta_0}{N} = \sqrt{\frac{\mu_0}{\varepsilon}}$$

Propagation direction (vector $\boldsymbol{k}$)

$\boldsymbol{k} = \boldsymbol{k}' - i\boldsymbol{k}''$

- $\boldsymbol{k}' = n\,\omega/c$, $\boldsymbol{k}'' = \kappa\,\omega/c$
- $\boldsymbol{k}'$ and $\boldsymbol{k}''$ can have different different directions:
  - ☞ $\boldsymbol{k}'\cdot\boldsymbol{r} = const$: constant phase planes
  - ☞ $\boldsymbol{k}''\cdot\boldsymbol{r} = const$: constant amplitude planes
- $\boldsymbol{k}''$: boundary conditions of an absorbing sample



Propagation direction (vector $\boldsymbol{S}$)

- Energy flow (Poynting vector $\boldsymbol{S}$)

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \wedge \mathbf{H} \rangle = \tfrac{1}{2}\,\mathrm{Re}\left\{\mathbf{E} \wedge \mathbf{H}^*\right\}$$

$$\langle \boldsymbol{S} \rangle = \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{\boldsymbol{E} \wedge \left(\boldsymbol{k}^* \wedge \boldsymbol{E}^*\right)\right\} = \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{\boldsymbol{k}^*\left(\boldsymbol{E}\cdot\boldsymbol{E}^*\right) - \underbrace{(\boldsymbol{k}^*\cdot\boldsymbol{E})}_{0}\boldsymbol{E}^*\right\} =$$

$$= \frac{1}{2\omega\mu_0}\mathrm{Re}\left\{\left(\boldsymbol{k}' - i\boldsymbol{k}''\right)\left(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*\right)\, e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}\right\} = \frac{\boldsymbol{k}'}{2\omega\mu_0}\left|\boldsymbol{E}_0\right|^2 e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}$$

- Propagation along z ($\boldsymbol{k}$ // z)

$$\left|\boldsymbol{S}\right| = \left(2\omega\mu_0\right)^{-1} n\frac{\omega}{c}\left|\boldsymbol{E}_0\right|^2 e^{-2\omega\kappa z/c} = \frac{n}{2\eta_0}\left|\boldsymbol{E}_0\right|^2 e^{-\alpha z}$$

$$\alpha(\omega) = \frac{2\omega\kappa}{c} = \frac{4\pi\kappa}{\lambda}$$

Optical pulses: dispersion

- Dispersion relation (non-absorbing medium):

$$k = \frac{\omega}{c} n(\omega)$$

- Optical pulse as a linear combination of eigenmodes:

$$\boldsymbol{E}(t,z) = \int \boldsymbol{E}_0(k)\, e^{i(\omega(k)t - kz)} dk = \int \boldsymbol{E}_0(\omega)\, e^{i(\omega t - k(\omega) z)} d\omega$$

- Mean frequency ω_0 and wave vector k_0: $\Delta\omega \ll \omega_0$, $\Delta k \ll k_0$:

$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_0 (k - k_0) + \frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_0 (k - k_0)^2 \quad = \quad \omega_0 + v_g (k - k_0) + \frac{\beta}{2}(k - k_0)^2$$

$$k(\omega) = k_0 + \left(\frac{dk}{d\omega}\right)_0 (\omega - \omega_0) + \frac{1}{2}\left(\frac{d^2 k}{d\omega^2}\right)_0 (\omega - \omega_0)^2 \quad = \quad k_0 + v_g^{-1} (\omega - \omega_0) + \frac{\psi}{2}(\omega - \omega_0)^2$$

$\psi = -\beta / v_g^3$ … group velocity dispersion (GVD)

Group velocity (first order effects)

- Propagation of a pulse

$$\boldsymbol{E}(t,z) = \int_{-\infty}^{\infty} \boldsymbol{E}_0(k)\, e^{i(\omega_0 t - k_0 z)} e^{i\left(v_g (k-k_0) t - (k-k_0) z\right)} dk = e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} \boldsymbol{E}_0(k)\, e^{i(v_g t - z)(k-k_0)} dk =$$

$$== e^{i(\omega_0 t - k_0 z)}\, \boldsymbol{E}_1(z - v_g t) = \underbrace{e^{-ik_0(z - vt)}}_{(A)}\, \underbrace{\boldsymbol{E}_1(z - v_g t)}_{(B)}$$

- (*A*): field oscillation with the central frequency

  wave front velocity: $v = \dfrac{\omega_0}{k_0} = \dfrac{c}{n(\omega)}$

- (*B*): propagation of envelope without deformation

  group velocity $v_g = \left.\dfrac{d\omega}{dk}\right|_{\omega_0}$

$$k = \frac{\omega}{c} n(\omega) \qquad /d/dk \qquad \Rightarrow \qquad v_g = \frac{c}{n + \omega\, dn/d\omega}$$

Optical pulses: energy flow

- Poynting vector

$$\left|\left\langle \boldsymbol{S}(z-v_g t)\right\rangle\right| = \frac{n}{2\eta_0}\left|\boldsymbol{E}_1(z-v_g t)\right|^2 e^{-2\omega\kappa z/c}$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

- Energy conservation law

$$e^{2\omega\kappa z/c}\frac{\partial\langle U\rangle}{\partial t} = \frac{\omega}{2}\varepsilon''\boldsymbol{E}_1\boldsymbol{E}_1^* + \frac{1}{2}\left(\frac{d(\omega\varepsilon')}{d\omega}\frac{\partial\boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^* + \varepsilon'\frac{\partial\boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*\right) =$$

$$= \omega n\kappa\,\varepsilon_0\boldsymbol{E}_1\boldsymbol{E}_1^* + \underbrace{\left(n^2 + \frac{\omega}{2}dn^2/d\omega\right)}_{n(n+\omega dn/d\omega)=nc/v_g}\varepsilon_0\frac{\partial\boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*$$

$$\nabla\cdot\langle\boldsymbol{S}\rangle = \frac{n}{2\eta_0}\frac{\partial}{\partial z}\left|\boldsymbol{E}_1(z-v_g t)\right|^2 e^{-2\omega\kappa z/c} =$$

$$= -\left(\frac{\omega n\kappa}{\eta_0 c}\boldsymbol{E}_1\cdot\boldsymbol{E}_1^* + \frac{n}{\eta_0 v_g}\frac{\partial\boldsymbol{E}_1}{\partial t}\cdot\boldsymbol{E}_1^*\right)e^{-2\omega\kappa z/c} = -\frac{\partial\langle U\rangle}{\partial t}$$

Group velocity dispersion GVD (second order effects)

- Main effect: pulse broadening in time
- This treatment is necessary for
  - ☞ Ultrashort pulse (sub-50 fs) propagation
  - ☞ Ultrashort pulse (ps or sub-ps) generation in lasers (multiple passes through dispersive media)
  - ☞ Short pulse (sub-ns) propagation in fibers (extremely long distances)

$$\omega(k) = \omega_0 + v_g(k - k_0) + \frac{\beta}{2}(k - k_0)^2$$

$$k(\omega) = k_0 + v_g^{-1}(\omega - \omega_0) + \frac{\psi}{2}(\omega - \omega_0)^2$$

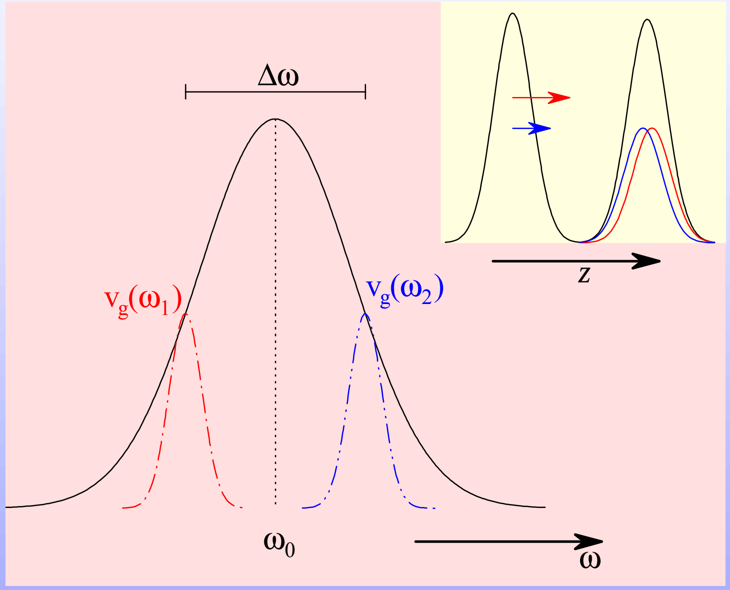
Intuitive treatment

- Spectral components coming from two ends of the spectrum propagate with different velocities

$$\Delta v_g = \frac{dv_g}{d\omega}\Delta\omega$$

- GVD is defined as

$$\psi = \frac{d^2k}{d\omega^2} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^2}\frac{dv_g}{d\omega}$$



- Spreading due to a thickness L of the dispersive material:

$$\Delta\tau \approx \frac{L}{v_g^2}\left|\Delta v_g\right| = \frac{L}{v_g^2}\left|\frac{dv_g}{d\omega}\right|\Delta\omega = L\,\psi\,\Delta\omega \quad \left(= 4\ln 2\,\frac{\psi L}{\tau_0}\right)$$

for a gaussian pulse shape

Exact treatment

- Superposition of the spectral components

$$\boldsymbol{E}(t,z)=\int_{-\infty}^{\infty}\boldsymbol{E}_0(\omega)\,e^{i(\omega t-k(\omega)z)}\,d\omega=e^{i(\omega_0 t-k_0 z)}\int_{-\infty}^{\infty}\boldsymbol{E}_0(\omega)\,e^{i(\omega-\omega_0)(t-z/v_g)}\;e^{i\beta z/(2v_g^3)(\omega-\omega_0)^2}\,d\omega$$

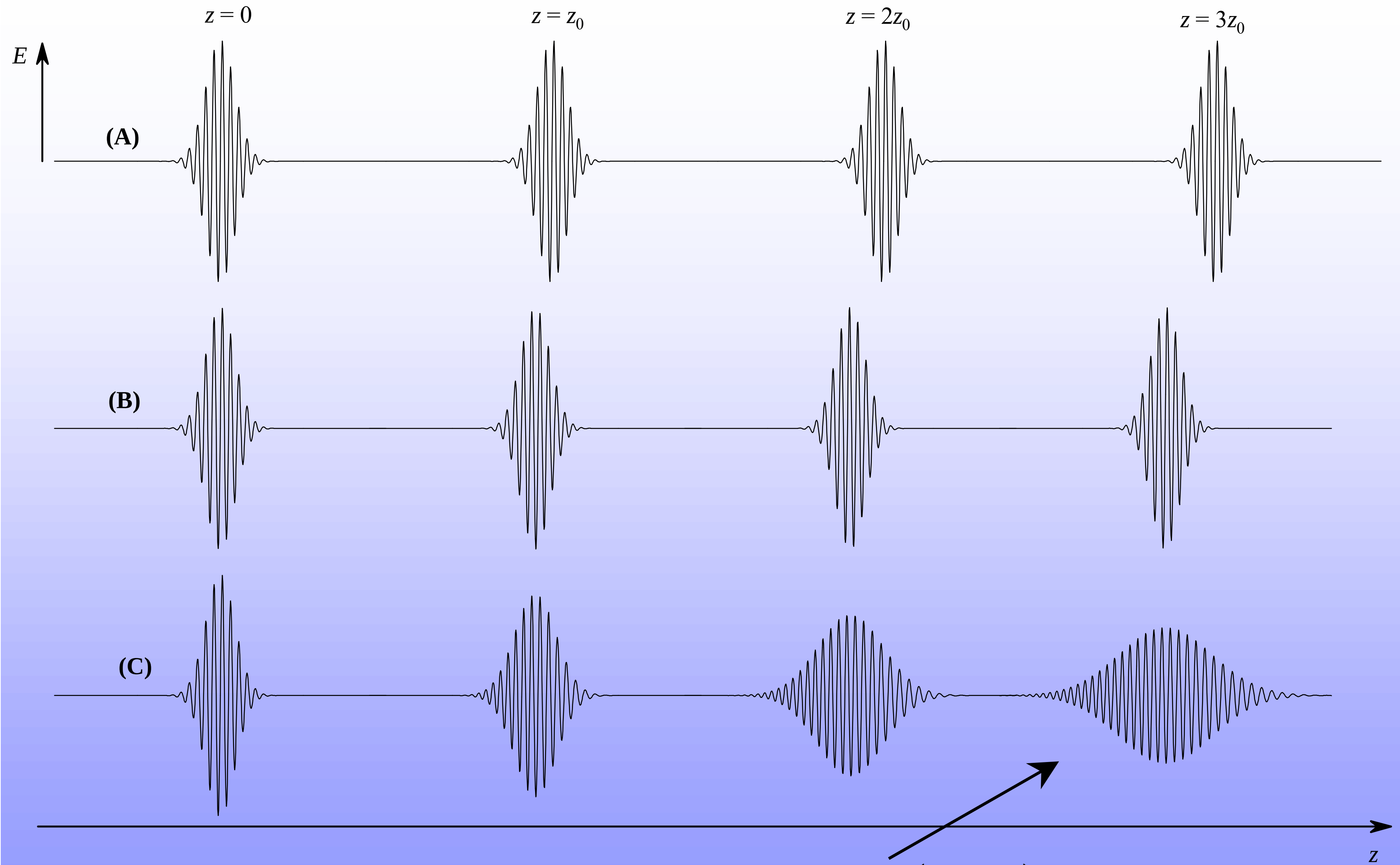
- One gets for a gaussian pulse for a crystal length *L* after all the integrations:

$$\mathbf{E}(t,z)=A e^{i(\omega_0 t-k_0 z)}\frac{1}{\sqrt{1+i\delta}}\;e^{-\frac{\alpha(t-z/v_g)^2(1-i\delta)}{(\delta^2+1)}}\qquad \text{with}\qquad \delta=2\alpha\psi L=\frac{4\ln 2\,\psi L}{\tau_0^2}$$

- Spreading due to a thickness *L* of the dispersive material:

$$\tau(L)=\tau_0\sqrt{1+\delta^2}=\tau_0\sqrt{1+\left(\frac{4\ln 2\,\psi L}{\tau_0^2}\right)^2}\qquad\left(\Delta\tau\approx\frac{4\ln 2\,\psi L}{\tau_0}\right)$$

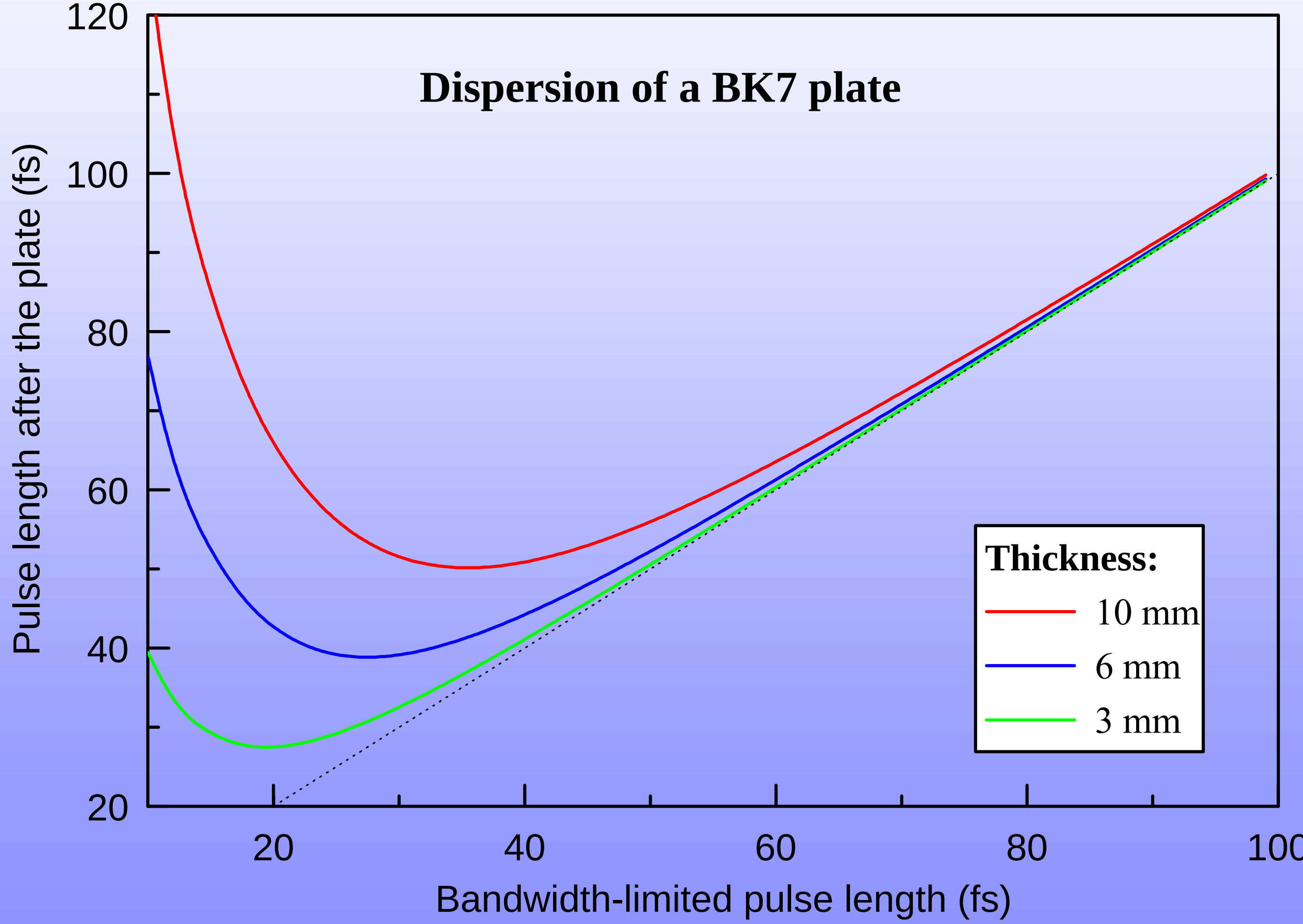
(intuitive treatment)



Mean frequency varies in time: $\omega(z,t) = \frac{\partial \Phi}{\partial t} = \omega_0 + \frac{(t - z/v_g)\,4\alpha^2\,\psi L}{1 + (2\alpha\,\psi L)^2}$

Chirp (group delay dispersion): $GDD = -\frac{\partial^2 \Phi}{\partial \omega^2} = \psi L$

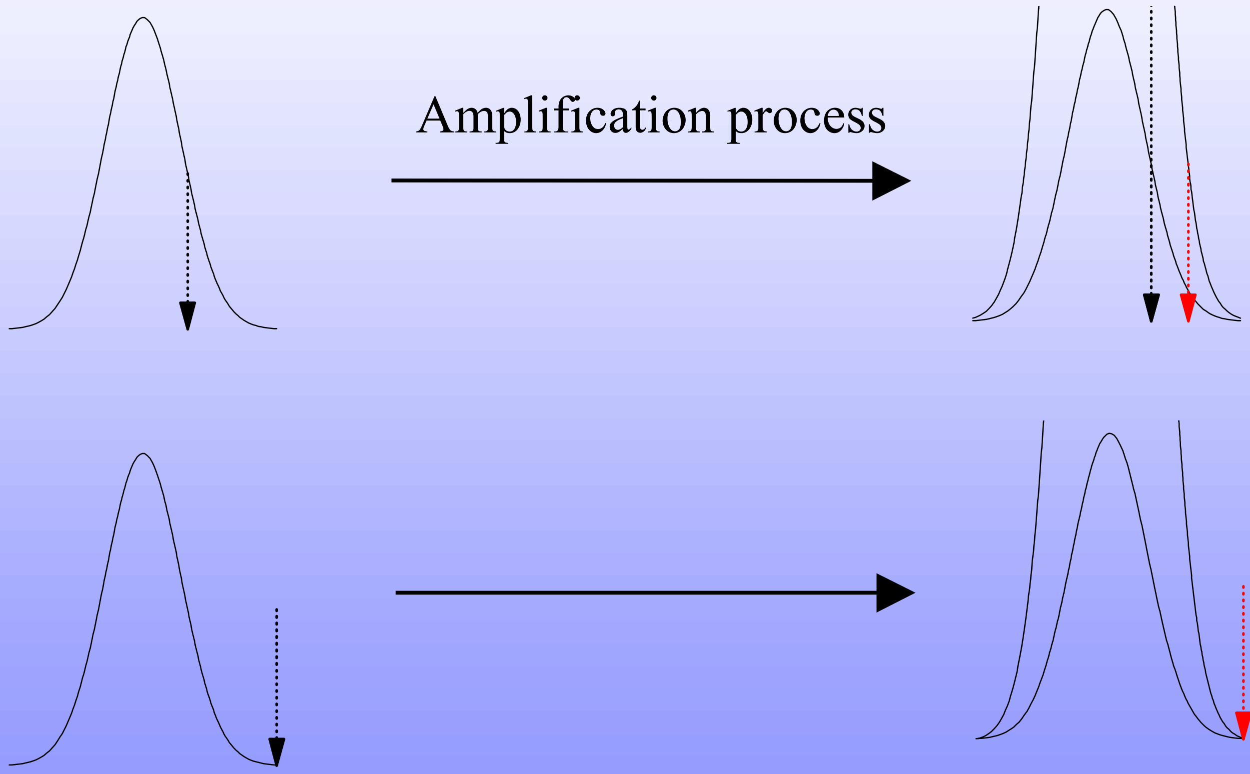
Pulse spreading: example



“Superluminal” effects

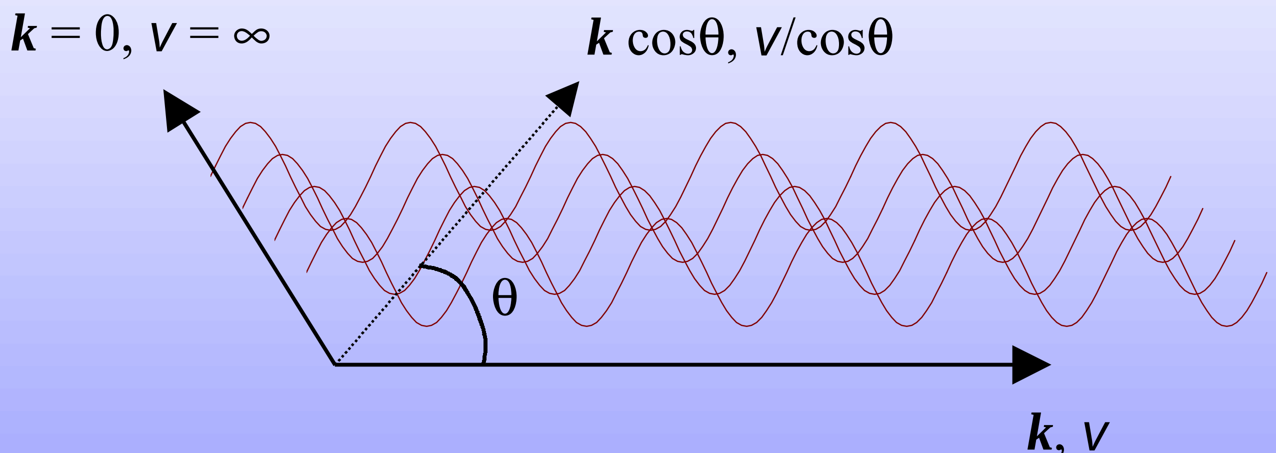
- What can be superluminal?
  - ☞ Phase velocity?
  - ☞ Group velocity?
  - ☞ Velocity of an information?
- Definition of the speed of an information transfer

Information speed



The noise can only slow down the information transfer

Phase velocity



Group velocity

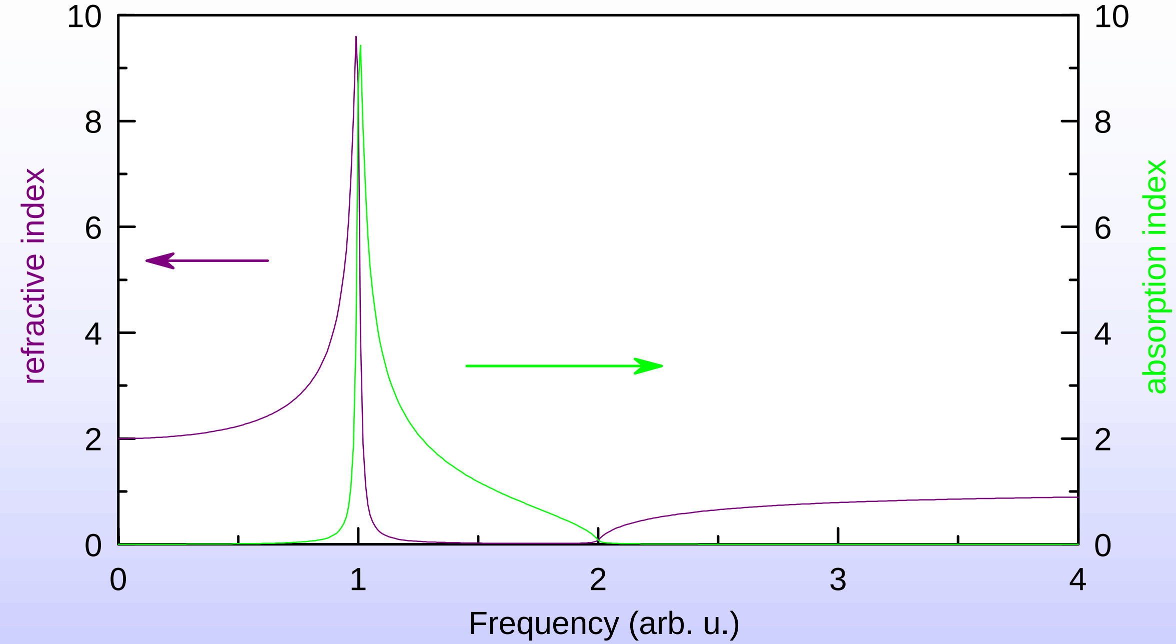
$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

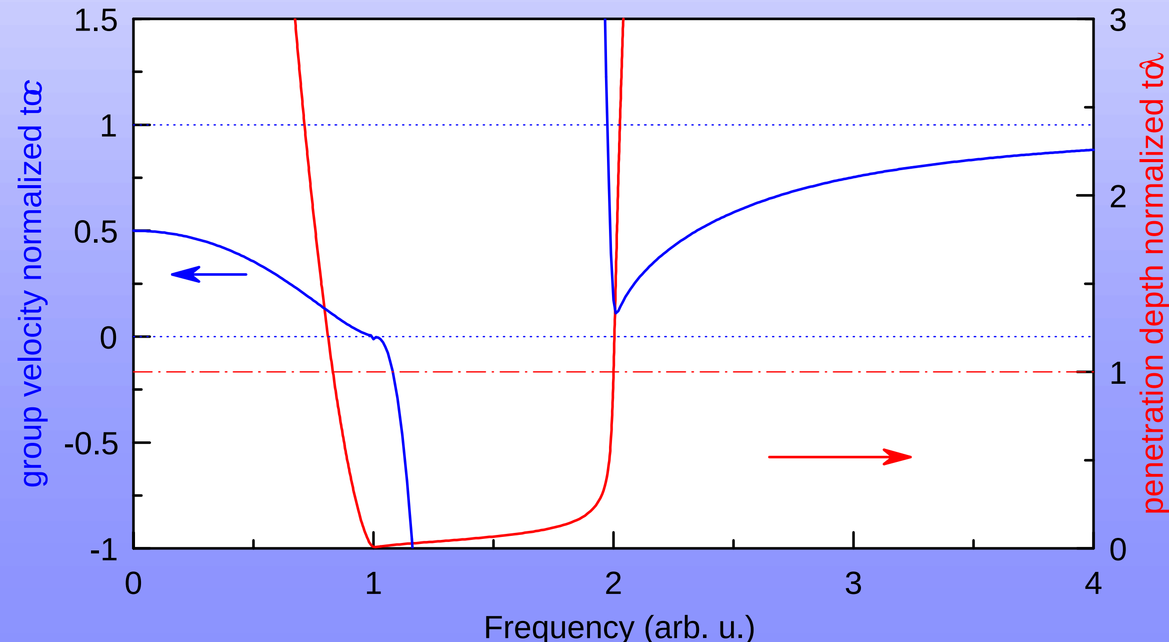
- Group velocity can exceed the vacuum speed of the light:
  - ☞ region of anomalous dispersion
    - dielectric resonance
    - gain medium

Dielectric resonance

- group velocity

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$





- group velocity can be superluminal
- penetration depth = fraction of λ

Gain-assisted superluminal light propagation

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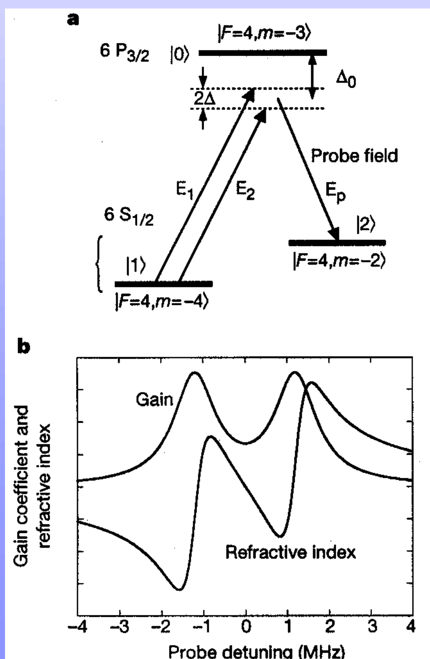
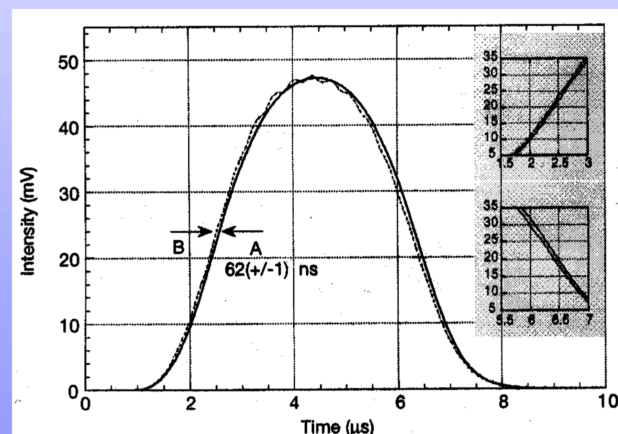
Gain-assisted anomalous dispersion:



Measured pulse advancement corresponds to a negative group velocity



Conclusion:

- pulse maximum and leading / trailing edge of the pulse are advanced
- there is *almost* (!!) no change in the pulse shape
- the very first non-zero signal can *never* (!!) be advanced