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Low temperature behavior of nonequilibrium multilevel systems

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Abstract

In this paper, we give a low temperature formula for the stationary occupations in Markovian systems away from detailed balance. Two applications are discussed, one to determine the direction of the ratchet current and one on population inversion. Both of these applications can take advantage of low temperatures to improve the gain and typical nonequilibrium features. Our new formula brings to the foreground the importance of kinetic aspects in terms of reactivities for deciding the levels with highest occupation and, thus, gives a detailed quantitative meaning to Landauer’s blowtorch theorem at low temperatures.

Keywords: nonequilibrium steady state, low temperature

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(Some figures may appear in colour only in the online journal)

1. Introduction

Multilevel systems and their dynamic characterization arise from different sources. On the one hand, they describe the weak coupling limit of small quantum systems in contact with an environment consisting of large equilibrium baths or radiation. When projecting the resulting open quantum dynamics on an energy basis, we get a jump process with rates that are determined by Fermi’s Golden Rule while the evolution follows the Pauli Master equation. The Markov limit (after the Born–Oppenheimer approximation) mostly excludes the effects of quantum coherence, although relaxation and nonequilibrium steady regimes can still be investigated [2, 5]. On the other hand, Markov jump processes also arise as effective dynamics on local minima of a thermodynamic potential from diffusion processes along the (Arrhenius–Eyring–)Kramers reaction rate theory [1, 3, 9]. The diffusion process itself derives from classical mechanics after suitable rescaling in a coarse-grained description.

These different derivations of Markov jump processes remain sometimes valid, in the case of driven dynamics, in the presence of nonequilibrium forces or when coupled to different reservoirs. In fact, conceptually, they often continue to play a major role in explorations of nonequilibrium physics and are used as models of nonequilibrium systems, they are even used to illustrate fundamental points in a nonequilibrium theory (e.g. of fluctuation and response). The exponential form of their rates mean that they are strongly nonlinear in potential differences and external forces, which is generally felt to be an important point in the study of far-from-equilibrium systems. More than diffusion processes, they also play a role in the modeling of chemical reactions and other aspects of life processes.

Very efficient computational tools exist for the solution of the Master equation, and over the last few decades a major part of their study has concentrated on numerical or simulation methods. There is, however, no systematic physical understanding of the solution of the Master equation away from detailed balance. In the present paper, we concentrate on the analytical theory of low temperature asymptotics in a steady nonequilibrium regime. We assume that the physical system under consideration is in contact with an environment at uniform temperature while also subject to external forcing that will break the Boltzmann occupation statistics. It is expected that nonequilibrium features can become more prominent at low temperatures. In addition, somewhat in analogy with equilibrium statistical mechanics, a more explicit asymptotic analysis is possible since there as a perturbation of the zero temperature phase diagram.

The present paper is a continuation of that program started in [6, 7]. The main formula in section 2 gives a low temperature expression for the occupation of a general multilevel system. The difference with [6] is that we now include the role of the reactivity or Arrhenius prefactors in the pre-asymptotic form. We give two applications where these reactivities matter, which are discussed in sections 3 and 4. The first is (a continuous time version of) Parrondo's game at low temperature [8], which is a (flashing potential) ratchet that is studied here for determining the direction of the ratchet current, this is a general problem that cannot be decided by entropic considerations alone. The second application concerns the role of equalizers in laser working. These equalizers contribute directly to the reactivities and are responsible for the necessary population inversion in laser working. In particular, we search for the influence of the equalizer in deciding the most occupied energy level, thus influencing the gain.

2. Low temperature formula

Low temperature analysis of nonequilibria requires us to find the analogue to the so-called ground states for statistical mechanical systems at equilibrium. The analysis in [6] did not distinguish between such dominant states and, therefore, a finite (low) temperature formula for the level occupations is needed. In this section, we propose the required extension of the Kirchhoff–Freidlin–Wentzel formula that was considered in [6]. The correction takes into account some of the kinetics of the model and will, therefore, also enable the study of (finite temperature) ‘noise’ effects that contribute to the nonequilibrium phenomenology.

The multilevel system is determined by a finite number of states $x, y, \dots \in K$, sometimes called energy levels because they often correspond to minima of some potential function or to the energy spectrum $E(x), E(y), \dots$ of a finite quantum system. (Ignoring possible degeneracies, we do not distinguish here between states and levels.)

We consider an irreducible continuous time Markov jump process on K with transition rates $\lambda(x, y) \equiv \lambda(x, y; \beta)$ for the jump $x \rightarrow y$ that depend on a parameter $\beta \geq 0$, which is interpreted as the inverse temperature. In the absence of other reservoirs

or of nonequilibrium forces, the dynamics satisfies the condition of detailed balance $\lambda(x, y)/\lambda(y, x) = \exp -\beta[E(y) - E(x)]$ for allowed transitions. The stationary occupation is then $\rho_{\text{eq}}(x) \propto \exp(-\beta E(x))$, giving the Boltzmann equilibrium statistics. Nonequilibrium conditions still preserve the dynamical reversibility $\lambda(x, y) \neq 0 \Rightarrow \lambda(y, x) \neq 0$ but a detailed balance is broken. Therefore, we are looking for a low temperature characterization of the nonequilibrium occupation.

After [6] we assume the existence of the limit

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \lambda(x, y) =: \phi(x, y) \tag{1}$$

which is abbreviated as $\lambda(x, y) \asymp e^{\beta\phi(x,y)}$. Similarly we define the log-asymptotic life time $\tau(x) \asymp e^{\beta\Gamma(x)}$ from the escape rates $\sum_y \lambda(x, y)$,

$$\Gamma(x) := - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \sum_y \lambda(x, y) = - \max_y \phi(x, y).$$

The log-asymptotic transition probability is then $e^{-\beta U(x,y)}$ with

$$U(x, y) := -\Gamma(x) - \phi(x, y).$$

The transition from x to y becomes more improbable as $U(x, y)$ grows larger. Clearly, $U(x, y) \geq 0$ and for all x there is at least one state $y \neq x$ for which $U(x, y) = 0$. All y for which $U(x, y) = 0$ are called preferred successors of x . This defines the digraph K^D with K as vertex set and having directed arcs (oriented edges) indicating all preferred successors.

We now define the reactivities $a(x, y)$, which will often be effective prefactors in a formula for reaction rates:

$$\lambda(x, y) = a(x, y) e^{-\beta[\Gamma(x)+U(x,y)]}, \quad a(x, y) := \lambda(x, y) e^{-\beta\phi(x,y)} = e^{o(\beta)} \tag{2}$$

as the sub-exponential part of $\lambda(x, y)$.

The new low temperature formula for the stationary distribution is $\rho(x)$, $x \in K$, which is a solution of the stationary Master equation $\sum_y [\lambda(x, y)\rho(y) - \lambda(y, x)\rho(x)] = 0$, $y \in K$. We derive this from the Kirchhoff formula on the graph \mathcal{G} , which has K as vertex set and with edges between any $x, y \in K$ where $\lambda(x, y) \neq 0$ ($\iff \lambda(y, x) \neq 0$) independent of β . In what follows, spanning trees \mathcal{T} refer to that graph \mathcal{G} , and \mathcal{T}_x denotes the in-tree to x defined for any tree \mathcal{T} by orienting every edge in \mathcal{T} towards x .

Theorem 2.1. *There is $\varepsilon > 0$ so that as $\beta \rightarrow \infty$,*

$$\rho(x) = \frac{1}{Z} A(x) e^{\beta[\Gamma(x)-\Theta(x)]} (1 + O(e^{-\beta\varepsilon})) \tag{3}$$

with

$$\Theta(x) := \min_{\mathcal{T}} U(\mathcal{T}_x) \quad \text{for} \quad U(\mathcal{T}_x) := \sum_{(y,y') \in \mathcal{T}_x} U(y, y') \quad \text{and} \tag{4}$$

$$A(x) := \sum_{\mathcal{T} \in M(x)} \prod_{(y,y') \in \mathcal{T}_x} a(y, y') = e^{o(\beta)} \tag{5}$$

where the last sum runs over all spanning trees, minimizing $U(\mathcal{T}_x)$ (i.e. $\mathcal{T} \in M(x)$ if $\Theta(x) = U(\mathcal{T}_x)$).

Before we give the proof at the end of this section, there are a number of remarks and illustrations that need to be made. The various terms in formula (3) have a physical interpretation in terms of lifetime Γ and accessibility Θ of states, cf [7] for an analysis of the boundary driven Kawasaki process. To be more specific, rewriting the rates in the

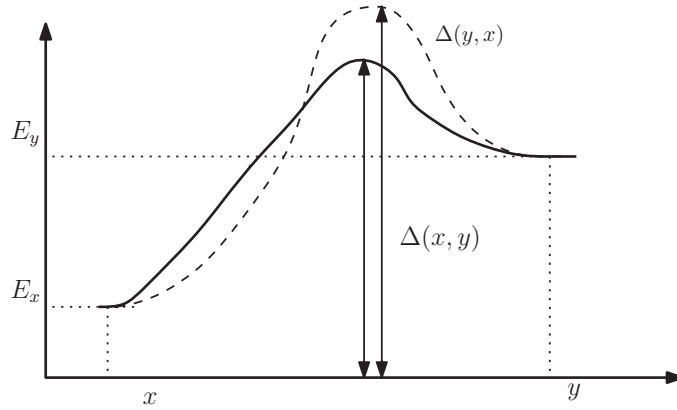


Figure 1. Climbing the barrier from x need not be the same as climbing it from y , as encoded in the work $\Delta(x, y) - \Delta(y, x)$.

Arrhenius form $\lambda(x, y) = a(x, y) \exp \beta[-\Delta(x, y) + E(x)]$ with $a(x, y) = a(y, x)$ shows that the symmetric prefactor has no exponential dependence on temperature. The energy $E(x)$ and barrier height $\Delta(x, y)$ refer to a thermodynamic potential landscape. Of course, this picture fails to a great extent when the system undergoes nonequilibrium driving and we should then think of the antisymmetric part in the barrier height $\Delta(x, y) - \Delta(y, x)$ as the work of non-conservative forces (see figure 1).

The local detailed balance form of the rates is

$$\begin{aligned} \lambda(x, y) &= \psi(x, y) e^{s(x,y)/2} \\ a(x, y) &:= \psi(x, y) e^{-\frac{\beta}{2}[E(x)+E(y)-\Delta(x,y)-\Delta(y,x)]}, \\ s(x, y) &:= \beta[E(x) - E(y) + \Delta(y, x) - \Delta(x, y)] \end{aligned} \tag{6}$$

where $s(x, y) = -s(y, x)$ is the entropy flux to the environment at inverse temperature β during $x \rightarrow y$. We see that the $a(x, y)$ truly pick up kinetic (non-thermodynamic) aspects of the transition rates. For (1) we have $\phi(x, y) = E(x) - \Delta(x, y)$. In the case of detailed balance, $\Delta(x, y) = \Delta(y, x)$, which implies that $\Theta(x)$ is a constant and $M(x)$ also does not depend on x making $A(x)$ constant. Nonequilibrium is responsible for having a state dependent set of minimizing trees. In other words, it is exactly for nonequilibrium that the reactivities start to matter in the occupation statistics. This is a version of Landauer’s blowtorch theorem, [4, 6]. We illustrate this with a simple model.

Example. Let us consider a nearest neighbor random walk on a ring $K = \{1, 2, \dots, N\}$, $N > 2$, where $N + 1 \equiv 1$. The transition rates are

$$\lambda(x, x + 1) = a_x e^{\frac{\beta}{2}q}, \quad \lambda(x + 1, x) = a_x e^{-\frac{\beta}{2}q}, \quad q > 0 \tag{7}$$

where $0 < a_x = e^{o(\beta)}$. The digraph is strongly connected; see figure 2.

We find that (4) implies that $\Theta(x) = 0$ for all states. Furthermore $\Gamma(x) = -q$, so that (3) gives

$$\frac{\rho(x)}{\rho(y)} = \frac{a_y}{a_x} + O(e^{-\beta\epsilon}) \tag{8}$$

and the most occupied states are naturally those that have lowest reactivities in the (for thermodynamic reasons) fastest direction (here: $x \rightarrow x + 1$) at low temperatures. The following two sections include more examples where the subasymptotic part $A(x)$ in (3) matters for nonequilibrium analysis.

Finally, (5) can be written as the determinant of an x -dependent matrix that is determined by putting weights $a(y, y')$ on the edges (y, y') of the digraph K^D , which is a consequence of the matrix-tree theorem [11]. The example in section 3 provides an illustration of this.

Proof of theorem 2.1 To find the stationary distribution we apply the Kirchoff formula on the graph \mathcal{G} :

$$\rho(x) = \frac{W(x)}{\sum_y W(y)}, \quad W(x) := \sum_{\mathcal{T}} w(\mathcal{T}_x) \tag{9}$$

in which the weight $w(\mathcal{T}_x)$ is the product of transition rates $\lambda(y, z)$ over all of the oriented edges (y, z) in the in-tree \mathcal{T}_x :

$$w(\mathcal{T}_x) := \prod_{(y,z) \in \mathcal{T}_x} \lambda(y, z). \tag{10}$$

We substitute (2):

$$\begin{aligned} W(x) &= \sum_{\mathcal{T}} \prod_{(y,z) \in \mathcal{T}_x} e^{-\beta(\Gamma(y)+U(y,z))} a(y, z) \\ &= \sum_{\mathcal{T}} \left(\prod_{y \neq x} e^{-\beta\Gamma(y)} \right) \prod_{(y,z) \in \mathcal{T}_x} e^{-\beta U(y,z)} a(y, z) \\ &= e^{\beta(\Gamma(x)-\Theta(x))-\beta \sum_y \Gamma(y)} \sum_{\mathcal{T}} e^{-\beta(U(\mathcal{T}_x)-\Theta(x))} \prod_{(y,z) \in \mathcal{T}_x} a(y, z), \end{aligned}$$

where $\Theta(x) = \min_{\mathcal{T}_x} U(\mathcal{T}_x)$. We split up the sum over spanning trees \mathcal{T} into two parts by collecting first the trees in $M(x)$, the set of trees minimizing $U(\mathcal{T}_x)$, and summing over the rest. We then continue

$$\begin{aligned} W(x) &= e^{\beta(\Gamma(x)-\Theta(x))-\beta \sum_y \Gamma(y)} \left(\sum_{\substack{\mathcal{T}_x \\ \mathcal{T} \in M(x)}} \prod_{(y,z) \in \mathcal{T}_x} a(y, z) \right. \\ &\quad \left. + \sum_{\substack{\mathcal{T}_x \\ \mathcal{T} \notin M(x)}} e^{-\beta(U(\mathcal{T}_x)-\Theta(x))} \prod_{(y,z) \in \mathcal{T}_x} a(y, z) \right) \\ &= e^{\beta(\Gamma(x)-\Theta(x))-\beta \sum_y \Gamma(y)} \sum_{\mathcal{T} \in M(x)} \prod_{(y,z) \in \mathcal{T}_x} a(y, z) (1 + O(e^{-\beta\varepsilon})) \end{aligned} \tag{11}$$

where $\varepsilon := \min_x \min_{\mathcal{T}, \mathcal{T} \notin M(x)} [U(\mathcal{T}_x) - \Theta(x)] > 0$. The normalization is

$$\sum_y W(y) = e^{-\beta \sum_z \Gamma(z)} \sum_y e^{\beta(\Gamma(y)-\Theta(y))} A(y) (1 + O(e^{-\beta\varepsilon}))$$

□

which ends the proof.

3. Low temperature ratchet current

We consider here a version of the well-known Parrondo game in continuous time and for random flipping between a flat potential (free random walker) and a nontrivial energy landscape, [8, 10]. We show how the formula (3) produces an expression for the low temperature ratchet current. In particular, its direction is not determined by entropic considerations alone but it also involves the reactivities.

The ratchet consists of two rings with the same number $N > 2$ of states. States are, therefore, denoted by $x = (i, n)$ where $i \in \{1 = N + 1, 2, \dots, N\}$ and $n = 0, 1$.

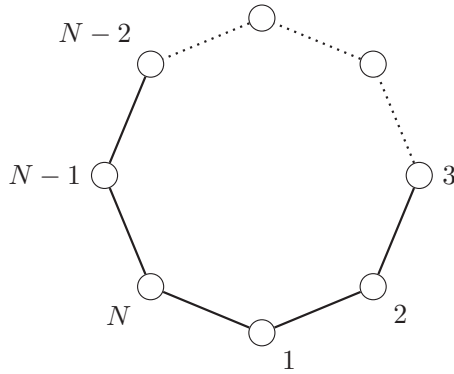


Figure 2. Digraph for the dynamics with rates (7). For any pair $\{x, y\}$ there is a connection $x \rightarrow y$.

We choose energies E_i with order $E_1 < \dots < E_N$. The transition rates on the outer ring ($n = 0$) are

$$\lambda((i, 0), (i + 1, 0)) = e^{\frac{\beta}{2}(E_i - E_{i+1})}, \quad \lambda((i + 1, 0), (i, 0)) = e^{\frac{\beta}{2}(E_{i+1} - E_i)}. \quad (12)$$

The rates on the inner ring ($n = 1$) are constant,

$$\lambda((i, 1), (i + 1, 1)) = \lambda((i + 1, 1), (i, 1)) = 1. \quad (13)$$

The rings are connected with transition rates $\lambda((i, n), (i, 1 - n)) = a$ for some $a > 0$. (Separately, that is, for $a = 0$, the dynamics on both rings satisfy detailed balance but with respect to another potential.) At low temperatures, the transitions $(i, 0) \rightarrow (i + 1, 0)$ become exponentially damped.

In a case when $a \ll 1$, the rings become uncoupled and the global detailed balance is restored for $a = 0$. On the other hand, when the coupling $a \gg 1$ becomes very strong, the model is effectively running on a single ring with transition rates being the sum

$$\lambda_{a=\infty}(i, i \pm 1) = e^{\frac{\beta}{2}(E_i - E_{i\pm 1})} + 1, \quad \frac{\lambda_\infty(i, i + 1)}{\lambda_\infty(i + 1, i)} = e^{\frac{\beta}{2}(E_i - E_{i+1})}$$

once again satisfying detailed balance but for inverse temperature $\beta/2$.

An interesting nonequilibrium situation arises when $a = 1$, see figure 3 for the graph of preferred transitions in this case. We apply formula (3) for the stationary distribution ρ . We have here $a(x, y) = 1$ over all edges. Therefore, in (5), the prefactor $A(x) = |M(x)|$ equals the number of in-trees for which $U(\mathcal{T}_x)$ is minimal. On the other hand, for the exponential factor in (3), $\Gamma(x) - \Theta(x)$ becomes maximal and equal to zero for $x \in \mathcal{D} := \{(1, 0), (i, 1), i = 1, \dots, N\}$; ($\Gamma(x) = 0 = \Theta(x)$ for $x \in \mathcal{D}$). Consequently, at low temperatures, $\rho(x) \propto |M(x)|$ for $x \in \mathcal{D}$, and $\rho(y) \simeq |M(y)| e^{\beta\Gamma(y)} / \mathcal{Z}$, $\Gamma(y) < 0$, is exponentially damped for $y \notin \mathcal{D}$. When the energy levels are equidistantly spaced in $[0, 1]$, then $\varepsilon \propto 1/N$ so that $\beta \gg N$ is required for theorem 2.1 to apply.

To discover the most occupied state, thus, amounts to finding $x \in \mathcal{D}$ with the largest $|M(x)|$, the number of in-trees for a given state x in the digraph. That number is given by the matrix-tree theorem (see, for example, [11]). We need the Laplacian matrix L on the digraph K^D and we erase the row and the column corresponding to vertex x to obtain the matrix L_x . Then,

$$A(x) = |M(x)| = \det L_x. \quad (14)$$

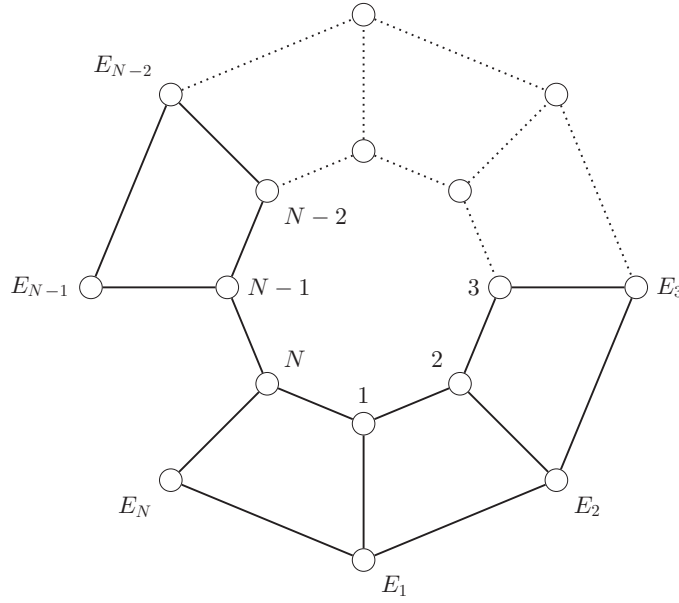


Figure 3. Digraph of preferred transitions of the ratchet system for the rates (12) on the outer and inner ring and where $a = 1$.

The Laplacian of the digraph K^D has a rather simple structure:

$$L = \begin{matrix} & \begin{matrix} (1, 0) & (2, 0) & \dots & \dots & (N, 0) & (1, 1) & (2, 1) & \dots & \dots & (N, 1) \end{matrix} \\ \begin{matrix} (1, 0) \\ (2, 0) \\ \vdots \\ \vdots \\ (N, 0) \\ (1, 1) \\ (2, 1) \\ \vdots \\ \vdots \\ (N, 1) \end{matrix} & \left(\begin{array}{cccccccccc} 1 & & & & & & -1 & & & \\ -1 & 1 & & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & & -1 & 1 & & & & \\ -1 & & & & 0 & 1 & & & & \\ -1 & & & & & & 3 & -1 & & -1 \\ & & -1 & & & & -1 & 3 & \ddots & \\ & & & \ddots & & & & \ddots & \ddots & \ddots \\ & & & & \ddots & & & & \ddots & \ddots & -1 \\ & & & & & -1 & -1 & & -1 & 3 \end{array} \right) \end{matrix}$$

The state for which the number of in-trees becomes maximal is $(1, 0)$. It is easy to see that there are more combinations to form an in-tree to $(1, 0)$ than to any other state $(i, 1)$ on the inner ring.

The stationary ratchet current J_R in the clockwise direction is the current over both rings, that is

$$J_R = j((i + 1, 0), (i, 0)) + j((i + 1, 1), (i, 1)) \tag{15}$$

with $j(x, y) = \lambda(x, y)\rho(x) - \lambda(y, x)\rho(y)$. Of course, the ratchet current also depends on N (ring size) and on the energy landscape. Interestingly, the direction of the ratchet current is not decided by the second law, in contrast with the usual case for currents produced by thermodynamic forces. For example, consider the following two trajectories

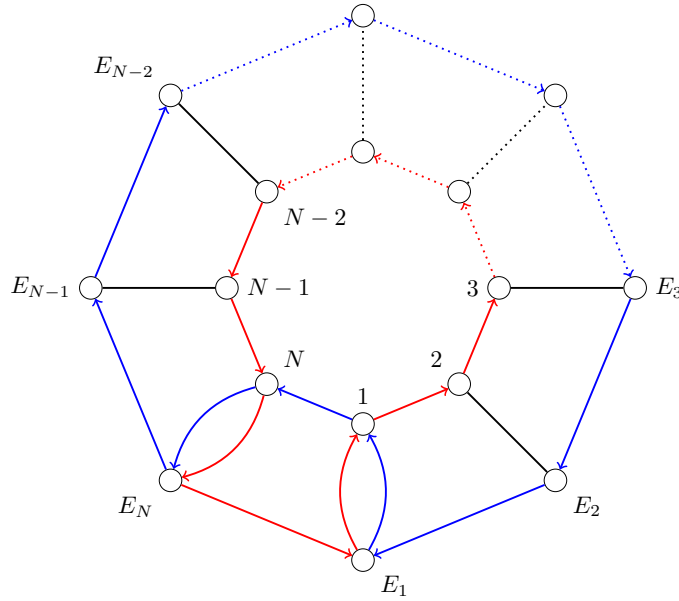


Figure 4. Trajectories ω_1 and ω_2 with the same entropy flux, yet in opposite directions.

to go around the ring: $\omega_1 = ((N, 0), (N - 1, 0), \dots, (1, 0), (1, 1), (N, 1), (N, 0))$ and $\omega_2 = ((N, 0), (1, 0), (1, 1), (2, 1), \dots, (N, 1), (N, 0))$. They wind in opposite directions while their entropy flux computed from (6) is exactly equal to $s(\omega_1) = s(\omega_2) = \beta(E_N - E_1) > 0$; figure 4.

This makes the point for kinetic aspects or, here, the importance of $A(x)$ in (5) for deciding the nature of the ratchet current.

Let us denote $x \simeq y$ when $x = y + O(e^{-\beta\epsilon})$. We take $x = (1, 1)$ for which $\rho(1, 1) \simeq \frac{1}{\mathcal{Z}}A((1, 1))$. Then,

$$j((2, 1), (1, 1)) \simeq \frac{1}{\mathcal{Z}}(A((2, 1)) - A((1, 1))).$$

Moreover,

$$j((2, 0), (1, 0)) \simeq \frac{A((2, 0))}{\mathcal{Z}}.$$

As a consequence,

$$J_R \simeq \frac{1}{\mathcal{Z}}(\det L_{(2,1)} + \det L_{(2,0)} - \det L_{(1,1)}).$$

From the form of the Laplacian L , one finds that

- (i) $\det L_{(2,0)} = 2 \det B_{N-1} - 3 \det B_{N-2} - 3$,
- (ii) $\det L_{(1,1)} = \det B_{N-1}$,
- (iii) $\det L_{(2,1)} = \det B_{N-2} + 1$

with

$$B_N = \begin{pmatrix} 3 & -1 & & \\ -1 & 3 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 3 \end{pmatrix}. \tag{16}$$

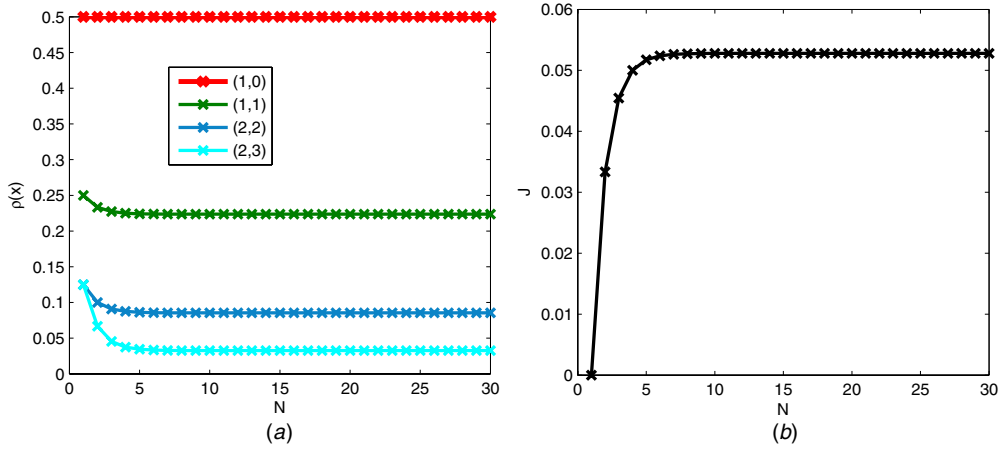


Figure 5. (a) Some occupations as a function of the ring size N . (b) The ratchet current J_R as a function of the size N from (17).

B_N satisfies the recursion relation $\det B_N = 3 \det B_{N-1} - \det B_{N-2}$, where $\det B_2 = 8$ and $\det B_1 = 3$. Hence,

$$\begin{aligned}
 J_R &\simeq \frac{\det B_{N-1} - 2 \det B_{N-2} - 2}{\mathcal{Z}} \\
 &= \frac{\det B_{N-2} - \det B_{N-3} - 2}{\mathcal{Z}}.
 \end{aligned}
 \tag{17}$$

Since $\det B_2 = 8$ and $\det B_1 = 3$, we have $\det B_2 > \det B_1 + 2$. Assume this holds for B_N , that is, that $\det B_N > \det B_{N-1} + 2$. Then,

$$\det B_{N+1} = 3 \det B_N - \det B_{N-1} > 3 \det B_N - \det B_{N-1} > 2 \det B_N - 2 > \det B_N - 2$$

which proves that $J_R > 0, \forall N \geq 4$ and $J_R = 0$ when $N = 3$. In other words, the direction is clockwise. The recursive relation allows us to calculate B_N for arbitrary N . The result for the ratchet current is shown in figure 5(b) (always up to order $O(e^{-\beta\epsilon})$). Note that the current saturates as the system size N increases due to the fact that the stationary occupations also saturate, see figure 5(a). This is, in turn, a consequence of the fact that the ratio between the number of in-trees for a specific state versus the sum of in-trees over all (dominant) states saturates.

4. Population inversion

Lasers provide another example of typical nonequilibrium phenomena. A laser medium consists of atoms with the same energy spectrum. One can, thus, treat the system as an ensemble of particles that populate different energy levels. The ensemble is in contact with an equilibrium heat reservoir (the lattice) at inverse temperature β . The nonequilibrium condition arises from an external field that excites particles to a higher energy level. The resulting population inversion is important for laser amplification.

Consider again a Markov jump process on $\{1, 2, \dots, N\}$ with $N > 2$ and define the transition rates as either purely relaxational or equalizing. Between any neighboring levels $x \leftrightarrow y, |x - y| = 1$, the relaxational dynamics follows detailed balance at inverse temperature β , and we take rates

$$\lambda(x, y) = \psi(x, y) e^{-\frac{\beta}{2}(E(y) - E(x))}, \quad \{x, y\} \neq \{1, N\}$$

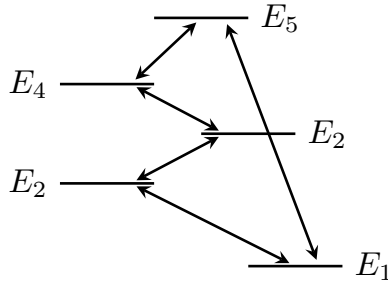


Figure 6. A five-level diagram, with the possible transitions.

for symmetric $\psi(x, y) = \psi(y, x)$ and energy levels $E(1) < E(2) < \dots < E(N)$, see figure 6 where $N = 3$, for example. There are also transitions between the lowest and the highest level in the form of an equalizing process

$$\lambda(1, N) = \lambda(N, 1) = b > 0$$

with b independent of temperature.

The question is, how to specify the low temperature stationary distribution? And, how to understand the role of the equalizer? In particular, we show how to organize the choice so that a specific level $i \neq 1$ gets most populated.

We take $\psi(i - 1, i) = \psi(i, i - 1) = a e^{-\beta F/2}$ for $F > 0$ and for some $a = e^{\alpha(\beta)}$. We also set $\psi(x, x + 1) = e^{\alpha(\beta)}$ for the other states $x \neq i - 1$. The digraph is strongly connected so that $\Theta(x) = 0$. From (3) the stationary occupations, thus, become $\rho(x) \simeq \frac{1}{Z} A(x) e^{\beta \Gamma(x)}$ and the lifetimes appear to be decisive. For $x = 1$ we have $\Gamma(1) = 0$. For $x = i$ it depends on: when $F \geq E_{i+1} - E_{i-1}$, then $2\Gamma(i) = E_{i+1} - E_i > 0$; and, when $E_i - E_{i-1} < F \leq E_{i+1} - E_{i-1}$, then $2\Gamma(i) = F - E_i + E_{i-1} > 0$ and the level i will be most occupied. However, when $F = E_i - E_{i-1}$, then from formula (3),

$$\begin{aligned} \rho(1) &\simeq \frac{\prod_{x \neq 1, i} \psi(x - 1, x)}{Z} a \\ \rho(i) &\simeq \frac{\prod_{x \neq 1, i} \psi(x - 1, x)}{Z} b \\ \rho(x) &\simeq 0, \quad x \neq \{1, i\}. \end{aligned} \tag{18}$$

The ratio a/b will determine whether level $x = 1$ or level $x = i$ will win the largest population.

Note that $b = 0$ would restore detailed balance and at that point no population inversion is possible. This again shows the important dependence of the reactivities for nonequilibrium statistics.

5. Conclusion and outlook

In contrast to thermal equilibrium, nonequilibrium steady states are also determined by kinetic aspects of the dynamics, which are encoded in different reactivities of stochastic transitions between states of the system. Some more insight and remarkable simplifications can be obtained in the low temperature regime, where the asymptotic behavior is related to dominant preferences within the Markovian transition rules.

In this paper, we have derived and discussed an improved asymptotic expression for the stationary occupations in nonequilibrium multilevel systems at low temperatures, explicitly including the β -sub-exponential corrections given by a restricted Kirchhoff tree formula. We

have illustrated its usefulness by evaluating the ratchet current in a model of a flashing potential, for which sub-exponential corrections play a crucial role by removing the ‘false’ degeneracies originating in a ‘naive’ exponential asymptotic analysis. This is a typical example of a problem where a purely thermodynamic approach based on the Second Law inequality does not suffice to determine the direction of a current. In addition, the phenomenon of population inversion is controlled by the variable transition reactivities (sometimes referred to as the Landauer blowtorch theorem), which we have demonstrated via our second example.

Although the geometric representations prove useful for the low temperature analysis of small stochastic systems, the complexity quickly increases with the number of degrees of freedom. It is still questionable if this approach can be extended to a useful framework for spatially extended systems with local transition rules. This could open a new route, for example, for studies of dynamical phase transitions in the low temperature domain.

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