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Spacetime expansions for weakly coupled interacting particle systems

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Abstract
We consider classes of both discrete time (parallel updating) and continuous time (sequential updating) interacting particle systems in the weak coupling regime. We set up a perturbation analysis for the spacetime distributions around the uncoupled dynamics and we construct the Gibbsian potential for the time-evolved measures.

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1. Introduction

Interacting particle systems are global Markov processes for spatially extended systems. With each site of a regular lattice, a (spin) variable with a finite number of possible values is associated. In the course of time, each spin is updated according to the (previous) values of itself and its neighbouring spins. The updating can be parallel in discrete time steps (the case of probabilistic cellular automata, PCA) or sequential in continuous time. The dynamics is fixed by giving the updating rule. That can be done in terms of rates (in continuous time) or in terms of transition probabilities (in discrete time). Initial data are specified in the form of a fixed initial spin configuration or, more generally, are given in terms of an initial probability measure $\mu = \mu_0$ on the spin configurations. In finite volume, extra boundary conditions must be obeyed. At a later time $t > 0$, the time-evolved measure μ_t is a solution of the associated Fokker–Planck equation. Basic references include [3, 6, 13]. Such dynamics are often used in studies of equilibrium and nonequilibrium systems but they are applied in an even wider variety of problems for simulation purposes.

We say that the interacting particle system is weakly coupled when the influence in updating from neighbouring spins is small. That means that we perturb around the case of independent updating where we have an infinite collection of uncoupled Markov chains. This should not be confused with the case where the system has transition rates (probabilities) with weak memory which could be called the regime of high noise. The difference is that, in

our case, the self-coupling of the spin to its previous state can be arbitrarily large. This will prove important in applications to continuous time models. After all, in continuous time the spin takes the same value for possibly very long times. In other words, the coupling in the 'vertical' temporal direction is strong whereas the 'horizontal' spatial coupling is sufficiently weak. In this respect, the present paper differs not only from the perturbation analysis in [10] (their theorem 1 in section 7.5, because we do not perturb around a spacetime product measure) but also from that in [2] (because we treat continuous time). For similar results, see [11][1].

The main question of the present paper is to identify sufficient conditions for which the time-evolved measure is Gibbsian for all (even infinite) times. In particular, is it possible to construct a uniformly absolutely summable interaction potential describing the time-evolved measure in the weak coupling regime? It is well documented that various quite natural transformations of Gibbs measures can give rise to non-Gibbsian measures, see e.g. [15]; ones for which no suitable sufficiently local interaction potential can be given except perhaps in the extended framework of weakly Gibbsian measures, see e.g. [8]. As such, it is not at all clear whether the transformation $\mu_0 \to \mu_t$ gives rise to a well-behaved effective potential for μ_t even when the dynamics is strictly local and μ_0 is Gibbsian. Note for example that in discrete time, for PCA with a local updating and when starting from a fixed configuration, at any fixed time t, any two spins that are sufficiently separated are in fact independent; it is far from obvious to imagine an interaction potential for μ_t that produces such an effect with immense cancellations for the correlation functions in the corresponding Gibbs distribution μ_t. We refer to [1] for an example where such a Gibbs measure is even Markovian, that is, it is associated with a nearest neighbour interaction.

On the other hand, one may be tempted to think that Gibbsianness is automatically obtained in the case of weak coupling. This is not correct: it is very much possible to find very weakly coupled dynamics and initial data for which the time-evolved measure is not Gibbsian for possibly an infinite time-interval. The easiest counterexample is that where the dynamics consists in fact of independent spin-flips (a so called infinite temperature Glauber dynamics) started from a low temperature phase of the standard ferromagnetic Ising model on the square lattice. As can be easily derived, see [14, 15], there is a finite time after which the time-evolved measure μ_t is no longer (ever) Gibbsian. In this way, the present paper is complementary to [14] and at the same time it gives a systematic treatment of various weak coupling spacetime expansions that are also used in [14]. The present paper is also an improvement on [9] where much of the same was attempted but containing important gaps both in formulation and in proofs. On the other hand, the set-up below gives a fully rigorous and detailed account of perturbation theory in the weak coupling regime of more general classes of interacting particle systems. The main result is a systematic description of the effective potential at any fixed time.

2. Plan

The paper is divided into two parts: section 3 deals with discrete time whereas section 4 treats the case of continuous time. They are not independent. The beginning of section 4 takes the continuous time limit of PCA. That is why the expansion for PCA in section 3 is done in such a way that it survives the limit where time becomes continuous. Moreover, many of the ideas and the techniques in the proofs of section 4 have analogues in discrete time.

[1] I Ignatiouk and V Malyshev have generalised these results for countable weakly interacting Markov chains by using the method of cluster expansion proposed by Pirogov and by the method of Lyapunov functions (1987).

The main results for PCA are collected in theorems 3.6 and 3.8. The first one deals with initial configurations that are fixed and the second one takes high temperature Gibbs measures as the initial condition. They both state the Gibbsian character of the time-evolved measure in the weak coupling regime. For continuous time, there is a first theorem 4.2 where the limit from PCA to spin-flip dynamics is discussed. The potential in continuous time is obtained as its limit for discrete time approximations. Finally, the more general result for continuous time models concerning Gibbsianness is contained in theorem 4.4. The proofs of the various theorems are each time obtained from combining various lemmas. As already mentioned in the introduction, the idea is to set up each time a spacetime expansion around the distribution obtained from the uncoupled dynamics. The simplest presentation of how this works can already be read in section 3.3.

3. Probabilistic cellular automata

A discrete time version of interacting particle systems is probabilistic cellular automata (PCA). These systems work with parallel updating rules. The weakly coupled dynamics gives rise to a spacetime distribution that is a small perturbation around a family of uncoupled one-dimensional lattice systems. Using the technique of cluster expansions, we make a perturbation expansion around that uncoupled system.

3.1. Notation

We restrict here only to models on the regular hyper-cubic lattice $\mathbb{Z}^d$ with the single site configuration space $S = \{-1, +1\}$. The configuration space of the model in a volume $\Lambda \subset \mathbb{Z}^d$ is $\Omega_\Lambda = S^\Lambda$ and we reserve the letters $\eta, \omega, \ldots$ to denote its elements. For any configuration $\eta \in \Omega_\Lambda$, we write $\eta(x)$ and $\eta(\Lambda')$ for the restriction of η to the site $x \in \Lambda$ and to the set $\Lambda' \subset \Lambda$, respectively. Further, we reserve the symbols $\eta_\bullet, \omega_\bullet, \ldots$ to denote the spacetime paths, i.e. the elements of the set $\Omega_\Lambda^{\mathbb{N}}$. Given a path $\eta_\bullet$, we write η_k for the configuration at time $k \in \mathbb{N}$. For any finite time $n \in \mathbb{N}$, we also introduce the shorthand $\Lambda^n = \Lambda \times \{1, \ldots, n\}$ for spacetime volumes and $\Omega_\Lambda^n = \Omega_\Lambda^{\{0,\ldots,n\}}$ for sets of finite time paths. If $\Lambda = \mathbb{Z}^d$, then the subscript specifying the volume will be omitted and we will only write $\Omega, \Omega^n, \ldots$. A set $\Lambda \subset \mathbb{Z}^d$ is called *connected* if it cannot be written as a union of two non-empty sets Λ_1 and Λ_2 such that $d(\Lambda_1, \Lambda_2) > 1$ in the metric $d(x, y) = \max_i |x_i - y_i|$. A function $f : \Omega \to \mathbb{R}$ is said to be local if there exists a finite set $D \subset \mathbb{Z}^d$ such that $f(\eta) = f(\eta')$ whenever $\eta(D) = \eta'(D)$. The smallest such D is called the *dependence set* and it is denoted by $\mathcal{D}_f$. We also use the symbol $\mathcal{L}$ for the set of local functions; note that $\mathcal{L}$ is a dense subset of the set of continuous functions $C(\Omega)$ in the uniform topology.

The PCA is a discrete time Markov process $\sigma_\bullet$ on Ω and we use the symbol $\mathbb{P}$ to denote its path-space measure. It is defined via transition probabilities $p_x(a \mid \eta)$, $x \in \mathbb{Z}^d$, of finding the spin $a \in S$ at the site x provided that the configuration at the previous time was η. The obvious conditions are $p_x(a \mid \eta) \geqslant 0$ and $\sum_{a \in S} p_x(a \mid \eta) = 1$ for every η. Moreover, we require the transition probabilities to be translation-invariant[2] and local. The latter means that there exists a finite set $B \subset \mathbb{Z}^d$, $|B| = b$, such that $p_x(a \mid \eta)$ depends only on the restriction $\eta(\tau_x(B))$ with τ_x being the shift map. At any time, all spins are simultaneously and independently updated, i.e. the process is fully introduced by the conditional probabilities, formally,

$$\mathbb{P}(\sigma_k = \eta' \mid \sigma_{k-1} = \eta) = \prod_{x \in \mathbb{Z}^d} p_x(\eta'(x) \mid \eta). \tag{3.1}$$

[2] The restriction to translation-invariant models only simplifies notation and it is not essential at all.

Indeed, the last formula together with an initial condition uniquely defines $\mathbb{P}$ since the process is to be Markovian. In particular,

$$\mathbb{P}(\sigma_n = \eta' \mid \sigma_0 = \eta) = \sum_{\substack{\eta_\bullet \in \Omega^n \\ \eta_0 = \eta, \eta_n = \eta'}} \prod_{k=1}^{n} \mathbb{P}(\sigma_k = \eta_k \mid \sigma_{k-1} = \eta_{k-1}). \tag{3.2}$$

We construct finite volume approximations of the process. For a finite set $\Lambda \subset \mathbb{Z}^d$, let a PCA be given as a Markov chain on Ω_Λ, via transition probabilities $p^\Lambda_{x,k}(a \mid \eta)$. As noted before, we also allow them to depend on time. The updating rules for the PCA in the volume Λ are

$$\mathbb{P}^\Lambda(\sigma_k = \eta' \mid \sigma_{k-1} = \eta) = \prod_{x \in \Lambda} p^\Lambda_{x,k}(\eta'(x) \mid \eta). \tag{3.3}$$

Similarly, in analogy with (3.2), the sum goes only over paths from the set Ω^n_Λ. Let Λ^* denote the set of all sites $x \in \Lambda$ such that $\tau_x(B) \subset \Lambda$. The PCA in the volume Λ will be called an *approximant* of the infinite volume process iff $p^\Lambda_{x,k}(a \mid \eta) = p_x(a \mid \eta)$ for any $x \in \Lambda^*$ and any $a \in S$, $\eta \in \Omega_\Lambda$. Note that this construction of finite-volume approximants covers a wide variety of boundary conditions, including fixed and free ones. A generalization also covering the periodic boundary conditions is straightforward.

3.2. Product dynamics

In order to study the PCA through a perturbation expansion, we need some reference process. For weak coupling, the most natural reference dynamics is provided by the product of 'single site' dynamics. We can use the formalism introduced in the last section and define it via the transition probability

$$p^0_x(a \mid \eta) = \mathsf{P}^0_{\eta(x),a} \tag{3.4}$$

where P^0 is a stochastic matrix on S. Its general form is

$$\mathsf{P}^0 = \begin{pmatrix} 1-\varepsilon_+ & \varepsilon_+ \\ \varepsilon_- & 1-\varepsilon_- \end{pmatrix} \qquad 0 \leqslant \varepsilon_\pm \leqslant 1. \tag{3.5}$$

Let us introduce the notation $2\varepsilon = \varepsilon_+ + \varepsilon_-$ and $\varepsilon_0 = \min\{\varepsilon_+, \varepsilon_-\}$. Obviously, the path-space measure now has the form $\mathbb{P}^0 = \bigotimes_{x \in \mathbb{Z}^d} \mathbb{P}^0_x$ and

$$\mathbb{P}^0_x(\sigma_n(x) = b \mid \sigma_0(x) = a) = (\mathsf{P}^0)^n_{ab}. \tag{3.6}$$

It is suitable to split the matrix P^0 into two parts,

$$\mathsf{P}^0 = \mathsf{H} + \mathsf{R} \tag{3.7}$$

where H is a stochastic matrix representing a 'no memory' process with the same invariant measure as P^0. So, it has the form $\mathsf{H}_{ab} = h_b$, where h is a normalized solution of the equation $\sum_a h_a \mathsf{P}^0_{ab} = h_b$. Under the assumption $\varepsilon > 0$, the invariant measure is unique and the matrix H is

$$\mathsf{H} = \frac{1}{2\varepsilon} \begin{pmatrix} \varepsilon_- & \varepsilon_+ \\ \varepsilon_- & \varepsilon_+ \end{pmatrix}. \tag{3.8}$$

The matrix H obviously satisfies the conditions $\mathsf{H}^n = \mathsf{H}$ and $\mathsf{H}\mathsf{P}^0 = \mathsf{H}$. Moreover, the orthogonality relations $\mathsf{H}\mathsf{R} = \mathsf{R}\mathsf{H} = 0$ hold true and, as a consequence, $(\mathsf{P}^0)^n = \mathsf{H} + \mathsf{R}^n$. By using the explicit formula

$$\mathsf{R}^n = \frac{1}{2\varepsilon} \begin{pmatrix} \varepsilon_+(1-2\varepsilon)^n & -\varepsilon_+(1-2\varepsilon)^n \\ -\varepsilon_-(1-2\varepsilon)^n & \varepsilon_-(1-2\varepsilon)^n \end{pmatrix} \tag{3.9}$$

we immediately obtain the following elementary relations which will prove useful in the study of the convergence of cluster expansions:

$$\min_{a,b\in S}(\mathsf{P}^0)^n_{ab} = \frac{\varepsilon_0}{2\varepsilon}[1-(1-2\varepsilon)^n] \tag{3.10}$$

and

$$\sum_{a\in S}\left|\mathsf{R}^n_{ab}\right| = (1-2\varepsilon)^n. \tag{3.11}$$

It is also useful to express the transition probabilities in the 'Gibbs form':

Lemma 3.1. *The transition probabilities are*

$$(\mathsf{P}^0)^n_{ab} = \frac{1}{z^n_a}\,\mathrm{e}^{(h+\delta h^n_a)b} \tag{3.12}$$

where

$$h = \frac{1}{2}\log\frac{\varepsilon_-}{\varepsilon_+} \qquad \delta h^n_a = \frac{a}{2}\log\frac{1+\mathrm{e}^{-2ha}(1-2\varepsilon)^n}{1-(1-2\varepsilon)^n} \tag{3.13}$$

and z^n_a is the normalization factor.

3.3. Perturbation expansion

We expand the general PCA in any volume Λ around a product dynamics and write the transition probability in the form

$$p^\Lambda_{x,k}(a\mid\eta) = p^0_x(a\mid\eta) + \beta^\Lambda_{x,k}(a\mid\eta). \tag{3.14}$$

Due to the properties of p^Λ and p^0, the perturbation β^Λ is local and satisfies the condition

$$\sum_{a\in S}\beta^\Lambda_{x,k}(a\mid\eta)\mathsf{H}_{ab} = h_b\sum_{a\in S}\beta^\Lambda_{x,k}(a\mid\eta) = 0. \tag{3.15}$$

We define the norm of β^Λ by

$$\|\beta^\Lambda\| = \sup_{\substack{a\in S\\ \eta\in\Omega_\Lambda}}\ \sup_{\substack{x\in\Lambda\\ k\in\mathbb{N}}}\left|\beta^\Lambda_{x,k}(a\mid\eta)\right|. \tag{3.16}$$

As we will see below, the expansion makes good sense provided that $\|\beta^\Lambda\| \ll \varepsilon_0$. Towards the end of this section, let an infinite-volume process be fixed and we deal with an approximant in a finite volume Λ.

Substituting (3.14) into (3.3) and introducing the simplified notation

$$\mathbb{P}^{\Lambda,n}_{\eta,\eta'} = \mathbb{P}^\Lambda(\sigma_n = \eta' \mid \sigma_0 = \eta) \tag{3.17}$$

we can subsequently write

$$\begin{aligned}\mathbb{P}^{\Lambda,n}_{\eta_0,\eta_n} &= \sum_{\eta_\bullet\in\Omega^n_\Lambda}\prod_{k=1}^{n}\prod_{x\in\Lambda}\left[p^0_x(\eta_k(x)\mid\eta_{k-1}) + \beta^\Lambda_{x,k}(\eta_k(x)\mid\eta_{k-1})\right]\\ &= \sum_{\Gamma\subset\Lambda^n}\sum_{\eta_\bullet\in\Omega^n_\Lambda}\prod_{(x,k)\in\Gamma}\beta^\Lambda_{x,k}(\eta_k(x)\mid\eta_{k-1})\prod_{(x,k)\in\Lambda^n\setminus\Gamma}\mathsf{P}^0_{\eta_{k-1}(x),\eta_k(x)}\\ &= \sum_{\Gamma\subset\Lambda^n}\varrho^{\Lambda,n}_{\eta_0,\eta_n}(\Gamma)\end{aligned} \tag{3.18}$$

where both sums over paths are restricted to fit the fixed configurations η_0, η_n. We call any subset Γ of the spacetime Λ^n a set of *interaction points* and we have introduced its

un-normalized weight, $\varrho^{\Lambda,n}_{\eta_0,\eta_n}(\Gamma) = \varrho(\Gamma)$, using the shorter notation whenever no confusion arises. We also define the *support*, $\underline{\Gamma}$, as the set of all sites $x \in \mathbb{Z}^d$ such that there exists a point $(x, k) \in \Gamma$. Further, the *set*, $\mathcal{P}(\Gamma)$, is defined by $\mathcal{P}(\Gamma) = \cup_{x\in\underline{\Gamma}} \tau_x(B)$.

Since for the reference process $(\mathbb{P}^0)^n_{\eta,\eta'} = \varrho^n_{\eta,\eta'}(\emptyset)$, we can relate the transition probabilities for both processes and write the perturbation expansion in the final form

$$\frac{\mathbb{P}^{\Lambda,n}_{\eta,\eta'}}{(\mathbb{P}^0)^n_{\eta,\eta'}} = \sum_{\Gamma\subset\Lambda^n} \bar{\varrho}^{\Lambda,n}_{\eta,\eta'}(\Gamma) \tag{3.19}$$

where the (*normalized*) *weight* of sets of interaction points is given by

$$\bar{\varrho}(\Gamma) = \frac{\varrho(\Gamma)}{\varrho(\emptyset)}. \tag{3.20}$$

If $\mathcal{P}(\Gamma) \subset \Lambda$, then it follows from (3.18) that $\bar{\varrho}^{\Lambda,n}_{\eta,\eta'}(\Gamma)$ does not depend on Λ and depends only on the restrictions $\eta(\mathcal{P}(\Gamma))$, $\eta'(\mathcal{P}(\Gamma))$. We define the set Γ to be *connected* iff we cannot write it as a union of two non-empty sets, $\Gamma = \Gamma_1 \cup \Gamma_2$, such that $\mathcal{P}(\Gamma_1) \cap \mathcal{P}(\Gamma_2) = \emptyset$. Any set Γ may be uniquely split into the family of its maximal connected components, $\Gamma = \{\gamma_i\}$, which we call *polymers*; we write $\mathcal{K}^n_\Lambda$ for the set of all polymers in the spacetime volume Λ^n. By using formulae (3.18) and (3.20), one can easily check the factorization property

$$\bar{\varrho}(\Gamma) = \prod_i \bar{\varrho}(\gamma_i) \tag{3.21}$$

where the product runs over all connected components of Γ. As a result, we have rewritten the LHS of (3.19) in the form of a standard polymer model, see e.g. [5] for details. Any set of polymers, $\mathcal{C}$, will be called a *cluster*, whenever it cannot be written as a union, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, of two non-empty sets such that $\mathcal{P}(\gamma_1) \cap \mathcal{P}(\gamma_2) = \emptyset$ for any $\gamma_1 \in \mathcal{C}_1$ and $\gamma_2 \in \mathcal{C}_2$. We introduce the symbol $\mathcal{P}(\mathcal{C}) = \cup_{\gamma\in\mathcal{C}}\mathcal{P}(\gamma)$ and use $\mathfrak{C}^n_\Lambda$ to denote the set of all clusters. Equation (3.19) may be rewritten in the form of the cluster expansion

$$\log \frac{\mathbb{P}^{\Lambda,n}_{\eta,\eta'}}{(\mathbb{P}^0)^n_{\eta,\eta'}} = \sum_{\mathcal{C}\in\mathfrak{C}^n_\Lambda} \Phi^{\Lambda,n}_{\eta,\eta'}(\mathcal{C}). \tag{3.22}$$

Our basic result collecting the properties of the cluster weights is the subject of the following lemma:

Lemma 3.2. *For any $a \geqslant 0$, there exists a constant $\tau_a > 0$ such that the following is true. If Λ is finite and the condition $\|\beta^\Lambda\| \leqslant \tau_a\varepsilon_0$ is satisfied, then*

(1) The cluster weights satisfy the bound

$$\sup_x \sup_n \sum_{\substack{\mathcal{C}\in\mathfrak{C}^n_\Lambda \\ x\in\mathcal{P}(\mathcal{C})}} \mathrm{e}^{a|\mathcal{P}(\mathcal{C})|} \sup_{\eta,\eta'} \left|\Phi^{\Lambda,n}_{\eta,\eta'}(\mathcal{C})\right| \leqslant 1. \tag{3.23}$$

(2) The weight $\Phi^{\Lambda,n}_{\eta,\eta'}(\mathcal{C})$ does not depend on Λ whenever $\mathcal{P}(\mathcal{C}) \subset \Lambda$. Further, it only depends on $\eta(\mathcal{P}(\mathcal{C}))$, $\eta'(\mathcal{P}(\mathcal{C}))$.

Proof. It follows from lemma 3.9 in section 3.6 by using a standard statement about the convergence of cluster expansion, see [5]. □

3.4. Fixed initial data

In this section, we study for weakly coupled PCA the Gibbsian structure of the marginal measures at each fixed time. We first establish the existence of a 'boundary condition independent' thermodynamic limit and then show the Gibbsianity of marginals at all times. The cluster expansions immediately provide us with an exponentially damped potential (in the diameter) for the marginal measures.

3.4.1. Thermodynamic limit. Given an approximant in a finite volume Λ, we introduce the symbol $\mu_\eta^{\Lambda,n}$ for the marginal of the path-space measure at time n. Let $f : \Omega \to \mathbb{R}$ be a local function with the *dependence set* $\mathcal{D}_f \subset \Lambda$. Then

$$\mu_\eta^{\Lambda,n}(f) = \sum_{\eta' \in \Omega_\Lambda} f(\eta')\mathbb{P}_{\eta,\eta'}^{\Lambda,n} = \sum_{\Gamma \subset \Lambda^n} \sum_{\eta' \in \Omega_\Lambda} f(\eta')\varrho_{\eta,\eta'}^{\Lambda,n}(\Gamma) \tag{3.24}$$

where we have used the polymer representation (3.18). For any set of interaction points Γ, we introduce the *root set* $\mathcal{R}(\Gamma)$ as the set of all points $(x, k) \in \Gamma$ for which there does not exist any $(y, l) \in \Gamma$ such that $l > k$ and $x \in \tau_y(B)$. Further, the symbol $\underline{\mathcal{R}(\Gamma)}$ denotes the support of $\mathcal{R}(\Gamma)$. We also use a natural generalization for clusters, $\mathcal{R}(\mathcal{C}) = \overline{\cup_{\Gamma \in \mathcal{C}} \mathcal{R}(\Gamma)}$.

We start with the following two lemmas. The first is an immediate consequence of equation (3.15) and the second follows from the definition of the cluster weights, see [5].

Lemma 3.3. *If $\underline{\mathcal{R}(\Gamma)} \not\subset \mathcal{D}_f$, then $\sum_{\eta' \in \Omega_\Lambda} f(\eta')\varrho_{\eta,\eta'}^{\Lambda,n}(\Gamma) = 0$.*

Lemma 3.4. $\sum_{\substack{\Gamma \subset \Lambda^n \\ \underline{\mathcal{R}(\Gamma)} \subset \mathcal{D}_f}} \bar{\varrho}_{\eta,\eta'}^{\Lambda,n}(\Gamma) = \exp\left[\sum_{\substack{\mathcal{C} \in \mathfrak{C}_\Lambda^n \\ \underline{\mathcal{R}(\mathcal{C})} \subset \mathcal{D}_f}} \Phi_{\eta,\eta'}^{\Lambda,n}(\mathcal{C})\right]$.

Using the above lemmas, we obtain the following cluster representation of expectations:

$$\begin{aligned}
\mu_\eta^{\Lambda,n}(f) &= \sum_{\substack{\Gamma \subset \Lambda^n \\ \underline{\mathcal{R}(\Gamma)} \subset \mathcal{D}_f}} \sum_{\eta' \in \Omega_\Lambda} f(\eta')\varrho_{\eta,\eta'}^{\Lambda,n}(\Gamma) \\
&= \sum_{\eta' \in \Omega_\Lambda} f(\eta')(\mathbb{P}^0)_{\eta,\eta'}^n \sum_{\substack{\Gamma \subset \Lambda^n \\ \underline{\mathcal{R}(\Gamma)} \subset \mathcal{D}_f}} \bar{\varrho}_{\eta,\eta'}^{\Lambda,n}(\Gamma) \\
&= \sum_{\eta' \in \Omega_\Lambda} f(\eta')(\mathbb{P}^0)_{\eta,\eta'}^n \exp\left(\sum_{\substack{\mathcal{C} \in \mathfrak{C}_\Lambda^n \\ \underline{\mathcal{R}(\mathcal{C})} \subset \mathcal{D}_f}} \Phi_{\eta,\eta'}^{\Lambda,n}(\mathcal{C})\right).
\end{aligned} \tag{3.25}$$

Given $\alpha \geqslant 1$, we say that the approximant in Λ is an *α-approximant* whenever the inequality $\|\beta^\Lambda\| \leqslant \alpha\|\beta\|$ is true.

Proposition 3.5. *Let $\|\beta\| \leqslant \alpha^{-1}\tau\varepsilon_0$, where τ may be chosen as a τ_a from lemma 3.2 for any $a > 0$. Then for any initial configuration $\eta \in \Omega$ there exist measures μ_η^n such that*[3]

$$\lim_\Lambda \sup_n \sup_\eta \left|\mu_\eta^{\Lambda,n}(f) - \mu_\eta^n(f)\right| = 0 \tag{3.26}$$

for any sequence of α-approximants and any local function $f : \Omega \to \mathbb{R}$.

[3] We use the symbols $\lim_\Lambda$ or $\lim_{\Lambda \uparrow \mathbb{Z}^d}$ for the limit along a sequence of finite volumes, $\{\Lambda_n\}_{n=1}^\infty$, such that any finite set $A \subset \mathbb{Z}^d$ is a subset of all Λ_n except for a finite number of them.

Proof. It is sufficient to prove that, for any local function $f : \Omega \to \mathbb{R}$, the sequence $\{\mu_\eta^{\Lambda,n}(f)\}$ is Cauchy, uniformly in n and η. Let A, B be finite sets of sites, $\mathcal{D}_f \subset A \subset B$. By using formula (3.25) we can write

$$\begin{aligned}
\left|\mu_\eta^{B,n}(f) - \mu_\eta^{A,n}(f)\right| &= \left|\sum_{\eta'\in\Omega_B} f(\eta')(\mathbb{P}^0)^n_{\eta,\eta'}\left[\exp\left(\sum_{\substack{\mathcal{C}\in\mathfrak{C}^n_B\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\Phi^{B,n}_{\eta,\eta'}(\mathcal{C})\right)\right.\right.\\
&\qquad\left.\left. - \exp\left(\sum_{\substack{\mathcal{C}\in\mathfrak{C}^n_A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right)\right]\right|\\
&\leqslant 2\|f\|\exp\left(\sup_{\eta,\eta'}\sum_{\substack{\mathcal{P}(\mathcal{C})\subset A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right|\right)\\
&\qquad\times\left(\sup_{\eta,\eta'}\sum_{\substack{\mathcal{P}(\mathcal{C})\not\subset A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right| + \sup_{\eta,\eta'}\sum_{\substack{\mathcal{P}(\mathcal{C})\not\subset A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\left|\Phi^{B,n}_{\eta,\eta'}(\mathcal{C})\right|\right).
\end{aligned} \tag{3.27}$$

To get the above inequalities, we first extended the configuration space to Ω_B for both expectations and then used lemma 3.2 to conclude $\Phi^{A,n}_{\eta,\eta'}(\mathcal{C}) = \Phi^{B,n}_{\eta,\eta'}(\mathcal{C})$ whenever $\mathcal{P}(\mathcal{C}) \subset A$. Finally, we used the inequality $|\mathrm{e}^x - \mathrm{e}^y| \leqslant 2(|x| + |y|)$ for $|x|, |y|$ small enough and the normalization of $\mathbb{P}^0$. By using lemma 3.2, we have the estimates

$$\sup_{\eta,\eta'}\sum_{\substack{\mathcal{P}(\mathcal{C})\subset A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right| \leqslant |\mathcal{D}_f|\sup_x\sup_{\eta,\eta'}\sum_{\substack{\mathcal{C}\in\mathfrak{C}^n_A\\ x\in\mathcal{P}(\mathcal{C})}}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right| \leqslant |\mathcal{D}_f| \tag{3.28}$$

and

$$\begin{aligned}
\sup_{\eta,\eta'}\sum_{\substack{\mathcal{P}(\mathcal{C})\not\subset A\\ \underline{\mathcal{R}(\mathcal{C})}\subset\mathcal{D}_f}}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right| &\leqslant |\mathcal{D}_f|\sup_{x\in\mathcal{D}_f}\sup_{\eta,\eta'}\sum_{x\in\mathcal{P}(\mathcal{C})\not\subset A}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right|\\
&\leqslant |\mathcal{D}_f|\exp\left(-a\inf_{\substack{M\text{ conn}\\ x\in M\not\subset A}}|M|\right)\sup_x\sup_{\eta,\eta'}\sum_{x\in\mathcal{P}(\mathcal{C})}\mathrm{e}^{a|\mathcal{P}(\mathcal{C})|}\left|\Phi^{A,n}_{\eta,\eta'}(\mathcal{C})\right|\\
&\leqslant |\mathcal{D}_f|\,\mathrm{e}^{-a\,\mathrm{dist}(\mathcal{D}_f,A^c)} \xrightarrow{A\uparrow\mathbb{Z}^d} 0
\end{aligned} \tag{3.29}$$

provided that $a > 0$ and $\|\beta^A\| \leqslant \tau_a\varepsilon_0$. Since the same argument can be used for the last sum in (3.27) as well, we obtain

$$\lim_{A,B\uparrow\mathbb{Z}^d}\sup_n\sup_\eta\left|\mu_\eta^{B,n}(f) - \mu_\eta^{A,n}(f)\right| = 0 \tag{3.30}$$

which finishes the proof. □

3.4.2. Potentials. The marginal measure $\mu_\eta^{\Lambda,n}$ of a Λ-approximant may be written in the Gibbs form

$$\mu_\eta^{\Lambda,n}(\eta') = \frac{1}{\mathcal{Z}_\eta^{\Lambda,n}}\mathrm{e}^{-H_\eta^{\Lambda,n}}(\eta') \tag{3.31}$$

where

$$H_\eta^{\Lambda,n}(\eta') = \sum_{A\subset\Lambda} U_\eta^{\Lambda,n}(A,\eta') \tag{3.32}$$

is the (finite-volume) Hamiltonian. The potential $U^\Lambda \equiv U_\eta^{\Lambda,n}$ may be split into two parts, the first, U^0, corresponding to the reference (single site) dynamics and the second, $\tilde{U}^\Lambda$, counting the interaction in. By using lemma 3.1 and formula (3.22), we immediately read

$$U_\eta^{0,n}(A,\eta') = \begin{cases} -\big(h + \delta h^n_{\eta(x)}\big)\eta'(x) & A = \{x\} \\ 0 & \text{otherwise} \end{cases} \tag{3.33}$$

and

$$\tilde{U}_\eta^{\Lambda,n}(A,\eta') = -\sum_{\substack{\mathcal{C}\in\mathfrak{C}^n_\Lambda \\ \mathcal{P}(\mathcal{C})\cap\Lambda = A}} \Phi^{\Lambda,n}_{\eta,\eta'}(\mathcal{C}). \tag{3.34}$$

Apart from the potentials corresponding to finite-volume approximants, we also define the (infinite-volume) potential $U = U^0 + \tilde{U}$, the interaction part of which is given by

$$\tilde{U}_\eta^n(A,\eta') = \tilde{U}_\eta^{\Lambda,n}(A,\eta') \tag{3.35}$$

for any Λ such that $A \subset \Lambda^*$. Lemma 3.2 assures that the above potential is well defined and uniformly bounded. Indeed, U^0 is clearly uniformly bounded for any $n > 0$ and inequality (3.23) implies

$$\sup_x \sup_n \sum_{A\ni x} \mathrm{e}^{a|A|} \sup_{\eta,\eta'} \big|\tilde{U}_\eta^n(A,\eta')\big| \leqslant 1. \tag{3.36}$$

The main goal of this section is to prove that the (infinite-volume) marginal μ_η^n is a Gibbs measure with respect to the potential U_η^n. Following a standard formalism (see [4]), we assign to any finite set of sites Δ the specification (from now on we omit the indices n, η), $\gamma_\Delta : C(\Omega) \to C(\Omega)$,[4] via

$$\gamma_\Delta f(\omega) = \frac{1}{\mathcal{Z}_\Delta^\omega} \sum_{\sigma\in\Omega_\Delta} \mathrm{e}^{-H_\Delta(\sigma\omega)} f(\sigma\omega). \tag{3.37}$$

Here we used the shorthand $\sigma\omega$ for the configuration $\sigma\omega(\Delta) = \sigma$, $\sigma\omega(\Delta^c) = \omega(\Delta^c)$, the symbol H_Δ for the Hamiltonian

$$H_\Delta(\sigma\omega) = \sum_{A\cap\Delta\neq\emptyset} U(A,\sigma\omega) \tag{3.38}$$

and $\mathcal{Z}_\Delta^\omega$ for the corresponding partition function with the boundary condition ω. A probability measure ν is said to be a Gibbs measure with respect to the potential U, whenever $\nu\gamma_\Delta(f) = \nu(f)$ for any Δ finite and $f \in C(\Omega)$.

Theorem 3.6. *Let the condition $\|\beta\| \leqslant \tau_a\varepsilon_0$ be true with τ_a being the constant from lemma 3.2 and $a > 0$. Then one has the following:*

(1) For any $\eta \in \Omega$ and $n > 0$, the marginal μ_η^n is a Gibbs measure with respect to the potential U_η^n.
(2) The interaction part of the potential satisfies

$$\sup_x \sup_n \sum_{A\ni x} \mathrm{e}^{a|A|} \sup_\eta \big\|\tilde{U}_\eta^n(A)\big\| \leqslant 1 \tag{3.39}$$

where we used the notation $\|\tilde{U}(A)\| = \sup_{\eta'} |\tilde{U}(A,\eta')|$.

[4] For uniformly bounded potentials the specifications are quasi-local, yielding $\gamma_\Delta f \in C(\Omega)$ for $f \in C(\Omega)$. See [4] for details.

Remark 3.7. Note that the potential fulfils $U^n_\eta(A) = 0$ whenever A is not a connected set. Since diam $A \leqslant |A| - 1$ for A connected, it follows from the second statement that the potential U^n_η is exponentially damped with the constant a.

Proof. Statement (2) is equation (3.36). To prove the first statement, we will proceed along the lines of [4]. We only need to establish the limit

$$\lim_\Lambda \left|\mu^\Lambda \gamma_\Delta(f) - \mu^\Lambda(f)\right| = 0 \tag{3.40}$$

for every local function f and Δ finite set of sites. Then the equality $\mu\gamma_\Delta(f) = \mu(f)$ immediately follows from proposition 3.5 and the continuity of $\gamma_\Delta f$ (see the note above), proving μ is Gibbsian w.r.t. U.

To prove (3.40), we first assign to any (finite-volume) potential U^Λ and $\Delta \subset \Lambda$ the Hamiltonian H^Λ_Δ and specification $\gamma^\Lambda_\Delta : C(\Omega_\Lambda) \to C(\Omega_\Lambda)$ via the obvious modification of (3.38) and (3.37). One can easily check the equality $\mu^\Lambda \gamma^\Lambda_\Delta = \mu^\Lambda$. Then,

$$\begin{aligned}
\left|\mu^\Lambda \gamma_\Delta(f) - \mu^\Lambda(f)\right| &= \left|\mu^\Lambda \left(\gamma_\Delta - \gamma^\Lambda_\Delta\right) f\right| \leqslant \left\|\left(\gamma_\Delta - \gamma^\Lambda_\Delta\right) f\right\| \\
&= \sup_{\omega\in\Omega} \left| \sum_{\xi\in\Omega_\Delta} \left(\frac{\mathrm{e}^{-H_\Delta(\xi\omega)}}{\mathcal{Z}^\omega_\Delta} - \frac{\mathrm{e}^{-H^\Lambda_\Delta(\xi\omega)}}{\mathcal{Z}^{\Lambda,\omega}_\Delta} \right) f(\xi\omega) \right| \\
&\leqslant \sup_{\omega\in\Omega} \sup_{\xi\in\Omega_\Delta} \left| \left(\frac{\mathcal{Z}^{\Lambda,\omega}_\Delta}{\mathcal{Z}^\omega_\Delta} \mathrm{e}^{(H^\Lambda_\Delta - H_\Delta)(\xi\omega)} - 1 \right) f(\xi\omega) \right| \\
&\leqslant \|f\| \left\| \frac{\mathcal{Z}^{\Lambda,\cdot}_\Delta}{\mathcal{Z}^\cdot_\Delta} \mathrm{e}^{(H^\Lambda_\Delta - H_\Delta)(\cdot)} - 1 \right\|.
\end{aligned}$$

It follows from statement (2) that

$$\begin{aligned}
\left\|H^\Lambda_\Delta - H_\Delta\right\| &\leqslant \sum_{\substack{A\cap\Delta\neq\emptyset \\ A\not\subset\Lambda^*}} \|U^\Lambda(A,\omega) - U(A,\omega)\| \\
&\leqslant |\Delta| \sup_{x\in\Delta} \sum_{\substack{A\ni x \\ A\not\subset\Lambda^*}} \left(\|U^\Lambda(A)\| + \|U(A)\|\right) \xrightarrow{\Lambda} 0.
\end{aligned} \tag{3.42}$$

Finally, using the inequality

$$\left\| \frac{\mathcal{Z}^{\Lambda,\cdot}_\Delta}{\mathcal{Z}^\cdot_\Delta} - 1 \right\| \leqslant \left\|\mathrm{e}^{H_\Delta - H^\Lambda_\Delta} - 1\right\| \leqslant \mathrm{e}^{\|H_\Delta - H^\Lambda_\Delta\|} - 1 \tag{3.43}$$

we immediately conclude that $\lim_\Lambda \left\|\mu^\Lambda \gamma_\Delta(f) - \mu^\Lambda(f)\right\| = 0$ as required. □

3.5. High-temperature initial data

In this section, we generalize the results for the fixed initial data to allow weakly correlated data. More precisely, we assume the initial condition to be a Gibbs measure corresponding to an interaction that is exponentially damped as the diameter of the interaction set grows, and show that theorem 3.6 keeps its validity.

Let $\lambda = \bigotimes_x \lambda_x$ be a product measure on Ω. We assume that the initial measure μ^0 is a Gibbs measure with the potential V and the *a priori* measure λ. By imposing free boundary conditions (it plays no role in the argument), the finite-volume approximation in Λ is given by

$$\mu^{\Lambda,0}(\mathrm{d}\eta) = \frac{1}{3^\Lambda} \lambda(\mathrm{d}\eta) \exp\left(- \sum_{A\subset\Lambda} V(A,\eta) \right). \tag{3.44}$$

Without making any restrictions, we will assume that $V(A) = 0$ whenever $|A| = 1$ or A is not a connected set. The marginal measure $\mu^{\Lambda,n}$ is then

$$\mu^{\Lambda,n}(\eta') = \int \mu^{\Lambda,0}(\mathrm{d}\eta)\mathbb{P}^{\Lambda,n}_{\eta,\eta'} = \frac{1}{3^{\Lambda}} \sum_{\Gamma\subset\Lambda^n} \int \lambda(\mathrm{d}\eta)\, \mathrm{e}^{-\sum_{A\subset\Lambda} V(A,\eta)} \varrho^{\Lambda,n}_{\eta,\eta'}(\Gamma). \tag{3.45}$$

Since we stick to the high-temperature regime, the Mayer expansion of the potential part proves useful and we can write

$$\mu^{\Lambda,n}(\eta') = \frac{1}{3^{\Lambda}} \sum_{\langle\Gamma,\mathfrak{A}\rangle} \int \lambda(\mathrm{d}\eta)\varrho^{\Lambda,n}_{\eta,\eta'}(\Gamma) \prod_{A\in\mathfrak{A}} \left(\mathrm{e}^{-V(A,\eta)} - 1\right) \tag{3.46}$$

where the symbol $\mathfrak{A}$ is used to denote a collection of sets and the sum runs over pairs of a set of interaction points and a collection of sets of sites. Introducing the notation

$$\nu^n_{\eta'}(\mathrm{d}\eta) = \frac{\lambda(\mathrm{d}\eta)(\mathbb{P}^0)^n_{\eta,\eta'}}{\int \lambda(\mathrm{d}\eta)(\mathbb{P}^0)^n_{\eta,\eta'}} \tag{3.47}$$

we define the weight of the pair $\langle\Gamma, \mathfrak{A}\rangle$ by

$$w^{\Lambda,n}_{\eta'}(\langle\Gamma,\mathfrak{A}\rangle) = \int \nu^n_{\eta'}(\mathrm{d}\eta)\bar{\varrho}^{\Lambda,n}_{\eta,\eta'}(\Gamma) \prod_{A\in\mathfrak{A}} \left(\mathrm{e}^{-V(A,\eta)} - 1\right). \tag{3.48}$$

Equation (3.46) may now be rewritten in the form

$$\mu^{\Lambda,n}(\eta') = \left(\frac{1}{3^{\Lambda}} \int \lambda(\mathrm{d}\eta)(\mathbb{P}^0)^n_{\eta,\eta'}\right) \sum_{\langle\Gamma,\mathfrak{A}\rangle} w^{\Lambda,n}_{\eta'}(\langle\Gamma,\mathfrak{A}\rangle) \tag{3.49}$$

which is a generalization of formula (3.19). All geometrical notions introduced for interaction sets may be naturally generalized to pairs. Given a pair $\langle\Gamma,\mathfrak{A}\rangle$, we define its support $\underline{\langle\Gamma,\mathfrak{A}\rangle} = \underline{\Gamma} \cup \bigcup_{A\in\mathfrak{A}} A$, the projection set $\mathcal{P}(\langle\Gamma,\mathfrak{A}\rangle) = \mathcal{P}(\Gamma) \cup \bigcup_{A\in\mathfrak{A}}$ and the root set $\mathcal{R}(\langle\Gamma,\mathfrak{A}\rangle) = \mathcal{R}(\Gamma)$. The pair $\langle\Gamma,\mathfrak{A}\rangle$ is called a *super-polymer* whenever it cannot be split into two pairs $\langle\Gamma_1,\mathfrak{A}_1\rangle$, $\langle\Gamma_2,\mathfrak{A}_2\rangle$ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$ and $\mathcal{P}(\langle\Gamma_1,\mathfrak{A}_1\rangle) \cap \mathcal{P}(\langle\Gamma_2,\mathfrak{A}_2\rangle) = \emptyset$. We denote the set of all super-polymers in the spacetime volume Λ^n by $\mathfrak{K}^n_{\Lambda}$. Obviously, any pair $\langle\Gamma,\mathfrak{A}\rangle$ may be viewed as a family of super-polymers $\{\langle\Gamma,\mathfrak{A}\rangle_i\}$ and the factorization of weights,

$$w^{\Lambda,n}_{\eta'}(\langle\Gamma,\mathfrak{A}\rangle) = \prod_i w^{\Lambda,n}_{\eta'}(\langle\Gamma,\mathfrak{A}\rangle_i) \tag{3.50}$$

takes place, explaining the above geometrical definitions. Further, the *super-clusters* are introduced in the obvious way and we reserve the symbol $\mathfrak{F}^n_{\Lambda}$ to denote the set of all super-clusters in Λ^n and generalize the geometrical notions to them in the natural way. Expanding the sum over super-polymers, equation (3.49) reads

$$\mu^{\Lambda,n}(\eta') = \frac{1}{3^{\Lambda}} \int \lambda(\mathrm{d}\eta)(\mathbb{P}^0)^n_{\eta,\eta'} \exp\left(\sum_{\mathcal{F}\in\mathfrak{F}^n_{\Lambda}} \Psi^{\Lambda,n}_{\eta'}(\mathcal{F})\right) \tag{3.51}$$

where $\Psi(\mathcal{F})$ is the weight of the super-cluster $\mathcal{F}$. The Gibbs form of marginal (3.31), (3.32) is established with the potential $U^{\Lambda} = U^{0,n} + \tilde{U}^{\Lambda,n}$. Here $U^{0,n}$ is the self-potential part corresponding to the reference dynamics and independent initial data, and the interaction part is

$$U^{\Lambda,n}(A,\eta') = -\sum_{\substack{\mathcal{F}\in\mathfrak{F}^n_{\Lambda}\\ \mathcal{P}(\mathcal{F})=A}} \Psi^{\Lambda,n}_{\eta'}(\mathcal{F}). \tag{3.52}$$

In the same way, as in the last section, we use the symbols U^n and $\tilde{U}^n$ for the Λ-independent potential and its interacting part, respectively.

Theorem 3.8. *Given $a > 0$, there exist constants τ'_a, $\upsilon_a > 0$ such that if the inequality*

$$\frac{\alpha\|\beta\|}{\tau'_a\varepsilon_0} + \sup_x \sum_{A\ni x} \mathrm{e}^{\upsilon_a|A|}\big(\mathrm{e}^{\|V(A)\|} - 1\big) \leqslant 1 \tag{3.53}$$

is true for some $\alpha \geqslant 1$, then we have the following for any $n > 0$:

(1) There is a measure μ^n such that $\lim_\Lambda \mu^{\Lambda,n} = \mu^n$ weakly for every sequence of α-approximants.
(2) The measure μ^n is Gibbsian w.r.t. the potential U^n.
(3) The interaction part of the potential fulfils

$$\sup_x \sum_{A\ni x} \mathrm{e}^{a|A|} \sup_n \|\tilde{U}^n(A)\| \leqslant 1. \tag{3.54}$$

Proof. It relies on the generalization of lemma 3.2 to super-polymers. Establishing the exponential damping of super-polymers, which is given in lemma 3.12, we can continue according lines of section 3.4 without essential changes. □

3.6. Geometry of polymers

In this section, we prove the basic statements about the convergence of cluster expansions formulated and exploited in last sections. Before that we need to extend a bit the notation introduced in sections 3.3 and 3.4. For any set of interaction points, $\Gamma \subset \Lambda^n$, we define the sets

$$B_\Gamma = \cup_{(x,k)\in\Gamma}(\tau_x(B) \cap \Lambda)\{k-1\} \tag{3.55}$$

and

$$B^*_\Gamma = B_\Gamma \backslash \{(x, k-1);\, (x,k) \in \Gamma\}. \tag{3.56}$$

Further, $\bar{\Gamma} = \Gamma \cup B_\Gamma$ is called the *dependence set* of Γ. Finally, for any spacetime point (x, k) we use the symbol $t_{x,k}$ for the largest integer $i \geqslant k$ such that $(x, j) \notin \bar{\Gamma}$ for any integer j satisfying $k < j < i$. Note that $t_{x,k} = n$ whenever (x, k) belongs to the root set $\mathcal{R}(\Gamma)$. By using these definitions and formula (3.15), we can sum out the spins in the set $\Lambda^n\backslash\bar{\Gamma}$ and write the unnormalized weight defined by (3.18) in the form

$$\varrho^{\Lambda,n}_{\eta_0,\eta_n}(\Gamma) = \sum_{\eta_\bullet(\bar{\Gamma})} \prod_{x\in\Lambda} (\mathsf{P}^0)^{t_{x,0}}_{\eta_0(x),\eta_{t_{x,0}}(x)} \prod_{(x,k)\in B^*_\Gamma} (\mathsf{P}^0)^{t_{x,k}-k}_{\eta_k(x),\eta_{t_{x,k}}(x)} \times \prod_{(x,k)\in\Gamma} \Big[\beta^\Lambda_{x,k}(\eta_k(x)\big|\eta_{k-1})\mathsf{R}^{t_{x,k}-k}_{\eta_k(x),\eta_{t_{x,k}}(x)}\Big] \tag{3.57}$$

the sum being taken over all configurations in the dependence set of Γ which are consistent with η_0, η_n. Note that, for any point $(x, k) \in \Gamma$, the 'free propagator' P^0 was replaced with R due to (3.15), which is exponentially damped. This is the key observation which enables us to find an estimate on polymer weights which is uniform in time. Namely, we state the following:

Lemma 3.9. *For any $a \geqslant 0$ there exists a constant $\tau_a > 0$ such that whenever the condition $\|\beta^\Lambda\| \leqslant \tau_a\varepsilon_0$ is satisfied, then*

$$\sup_x \sup_n \sum_{\substack{\gamma\in\mathcal{K}^n_\Lambda \\ x\in\mathcal{P}(\gamma)}} \mathrm{e}^{(1+a)|\mathcal{P}(\gamma)|} \sup_{\eta,\eta'} \big|\bar{\varrho}^{\Lambda,n}_{\eta,\eta'}\big| \leqslant 1. \tag{3.58}$$

Proof of lemma 3.9. In order to prove the lemma, we need to introduce a suitable geometrical representation of polymers which is done in three steps.

Step 1: On any polymer γ we build the directed graph $\mathcal{G}(\gamma)$ defined in such a way that the set of vertices of $\mathcal{G}(\gamma)$ is γ and the vertex $(x,k) \in \gamma$ *points to* the vertex $(y,l) \in \gamma$ iff $(x, t_{x,k}) \in B_{\{(y,l)\}}$. It is obvious that any vertex points to at most one other vertex and the root set is exactly the set of sites for which there is none. Since γ is a finite set, the root set cannot be empty. Any polymer γ with just one root is called *simple* and it is easy to realize that the graph $\mathcal{G}(\gamma)$ of any simple polymer γ is a tree-graph. Any polymer γ may be uniquely written as a disjoint union of simple polymers; let us write $\gamma = \{\gamma_\alpha\}$. Note that the family of sets $\{\mathcal{P}(\gamma_\alpha)\}$ is a cluster. A set $\{\gamma_\alpha\}$ of simple polymers is called *compatible* iff there is a polymer γ such that $\{\gamma_\alpha\}$ is the family of its simple parts.

Step 2: We say that the simple polymers γ_1 and γ_2 are equivalent if there exists an isomorphism E between the directed graphs $\mathcal{G}(\gamma_1)$ and $\mathcal{G}(\gamma_2)$ such that $E(x,k) = (y,l)$ implies $x = y$ for any vertices $(x,k) \in \gamma_1$ and $(y,l) \in \gamma_2$. It means that equivalent simple polymers can differ only in time coordinates of interaction points. The classes of equivalent simple polymers will be called *skeletons* and we will reserve the symbols $\mathfrak{S}, \mathfrak{S}_1, \ldots$ for them. Since the projection set $\mathcal{P}(\gamma)$ is the same for all equivalent simple polymers, we can naturally introduce the symbol $\mathcal{P}(\mathfrak{S})$ for the projection of the skeleton $\mathfrak{S}$ and, similarly, $\mathcal{R}(\mathfrak{S})$ for the root set.

Step 3: A collection of skeletons $\mathfrak{S} = \{\mathfrak{S}_\alpha\}$ is called a cluster whenever the corresponding collection $\{\mathcal{P}(\mathfrak{S}_\alpha)\}$ of sets is a cluster. We assign to it a graph of connectivity, $\mathcal{H}(\mathfrak{S})$, in the sense that the skeletons $\mathfrak{S}_1, \mathfrak{S}_2$ are connected by edge iff $\mathcal{P}(\mathfrak{S}_1)\mathcal{P} \cap (\mathfrak{S}_2) \neq \emptyset$. Since $\mathfrak{S}$ is a cluster, the graph $\mathcal{H}(\mathfrak{S})$ is clearly connected.

By using equations (3.57) and (3.11) and the normalization condition $\sum_{b\in S}(\mathsf{P}^0)^n_{ab} = 1$, it is not hard to realize that the following upper bound for the unnormalized weights holds true:

$$\begin{aligned}
\left|\varrho^{\Lambda,n}_{\eta,\eta'}(\gamma)\right| &\leqslant \prod_{(x,k)\in\mathcal{R}(\gamma)} \left(\|\beta^\Lambda\| \sum_{a\in S} \left|\mathsf{R}^{n-k}_{a,\eta_n(x)}\right| \right) \\
&\quad \times \prod_{(x,k)\in\gamma\setminus\mathcal{R}(\gamma)} \left(\|\beta^\Lambda\| \sum_{a,b\in S} \left|\mathsf{R}^{t_{x,k}-k}_{ab}\right| \right) \prod_{x\in\Lambda\setminus\underline{\gamma}} (\mathsf{P}^0)^n_{\eta_0(x),\eta_n(x)} \\
&\leqslant (2\|\beta^\Lambda\|)^{|\gamma|} \prod_{(x,k)\in\gamma} (1-2\varepsilon)^{t_{x,k}-k} \prod_{x\in\Lambda\setminus\underline{\gamma}} (\mathsf{P}^0)^n_{\eta_0(x),\eta_n(x)}.
\end{aligned} \tag{3.59}$$

A lower bound for the reference process is, due to (3.10),

$$\begin{aligned}
\varrho^{\Lambda,n}_{\eta,\eta'}(\emptyset) &= \prod_{x\in\underline{\gamma}} (\mathsf{P}^0)^n_{\eta_0(x),\eta_n(x)} \prod_{x\in\Lambda\setminus\underline{\gamma}} (\mathsf{P}^0)^n_{\eta_0(x),\eta_n(x)} \\
&\geqslant \left[\frac{\varepsilon_0}{2\varepsilon}(1-(1-2\varepsilon)^n)\right]^{|\underline{\gamma}|} \prod_{x\in\Lambda\setminus\underline{\gamma}} (\mathsf{P}^0)^n_{\eta_0(x),\eta_n(x)}.
\end{aligned} \tag{3.60}$$

Therefore, we obtain the estimate for the (normalized) weight

$$\begin{aligned}
\left|\bar{\varrho}^{\Lambda,n}_{\eta,\eta'}(\gamma)\right| &\leqslant \left[\frac{\varepsilon_0}{2\varepsilon}(1-(1-2\varepsilon)^n)\right]^{-|\underline{\gamma}|} (2\|\beta^\Lambda\|)^{|\gamma|} \prod_{(x,k)\in\gamma} (1-2\varepsilon)^{t_{x,k}-k} \\
&\leqslant \prod_\alpha w^n(\gamma_\alpha)
\end{aligned} \tag{3.61}$$

where the last product runs over all simple parts of γ and we have defined

$$w^n(\gamma_\alpha) = \left[\frac{4\varepsilon}{1-(1-2\varepsilon)^n}\frac{\|\beta^\Lambda\|}{\varepsilon_0}\right]^{|\gamma_\alpha|} \prod_{(x,k)\in\gamma_\alpha} (1-2\varepsilon)^{t_{x,k}-k}. \tag{3.62}$$

So, we have restored the factorization of the polymer weights into its simple parts at least for the above upper bound.

Going back to condition (3.58), we can use the above representation of polymers and estimate the left-hand side as follows:

$$\begin{aligned}\sum_{\substack{\gamma\in\mathcal{K}^n_\Lambda\\ x\in\mathcal{P}(\gamma)}} \mathrm{e}^{(1+a)|\mathcal{P}(\gamma)|}\sup_{\eta,\eta'}\left|\bar{\varrho}^n_{\eta,\eta'}(\gamma)\right| &\leqslant \sum_{\substack{\{\gamma_\alpha\}\,\mathrm{comp}\\ x\in\bigcup_\alpha\mathcal{P}(\gamma_\alpha)}} \prod_\alpha\left[\mathrm{e}^{(1+a)|\mathcal{P}(\gamma_\alpha)|}w^n(\gamma_\alpha)\right]\\ &= \sum_{\substack{\{\mathfrak{S}_\alpha\}\,\mathrm{cluster}\\ x\in\bigcup_\alpha\mathcal{P}(\mathfrak{S}_\alpha)}} \sum_{\substack{\{\gamma_\alpha\}\,\mathrm{comp}\\ \forall\alpha:\gamma_\alpha\in\mathfrak{S}_\alpha}} \prod_\alpha\left[\mathrm{e}^{(1+a)|\mathcal{P}(\gamma_\alpha)|}w^n(\gamma_\alpha)\right]\\ &\leqslant \sum_{\substack{\{\mathfrak{S}_\alpha\}\\ x\in\bigcup_\alpha\mathcal{P}(\mathfrak{S}_\alpha)}} \prod_\alpha\left[\mathrm{e}^{(1+a)|\mathcal{P}(\mathfrak{S}_\alpha)|}\sum_{\gamma_\alpha\in\mathfrak{S}_\alpha} w^n(\gamma_\alpha)\right]\end{aligned} \tag{3.63}$$

where in the last inequality we estimated the sum over all compatible simple polymers by omitting the condition of compatibility. To estimate the last sum, we can use (3.62) and write

$$\begin{aligned}\sum_{\gamma_\alpha\in\mathfrak{S}_\alpha} w^n(\gamma_\alpha) &= \left[\frac{4\varepsilon}{1-(1-2\varepsilon)^n}\frac{\|\beta^\Lambda\|}{\varepsilon_0}\right]^{|\mathfrak{S}_\alpha|}\sum_{\gamma_\alpha\in\mathfrak{S}_\alpha}\prod_{(x,k)\in\gamma_\alpha}(1-2\varepsilon)^{t_{x,k}-k}\\ &\leqslant \left[\frac{4\varepsilon}{1-(1-2\varepsilon)^n}\frac{\|\beta^\Lambda\|}{\varepsilon_0}\right]^{|\mathfrak{S}_\alpha|}\left[\sum_{l=0}^{n-1}(1-2\varepsilon)^l\right]^{|\mathfrak{S}_\alpha|} \leqslant (2\tau_a)^{|\mathfrak{S}_\alpha|}.\end{aligned} \tag{3.64}$$

since $\|\beta^\Lambda\| \leqslant \tau_a\varepsilon_0$ by assumption. Note that the above upper bound does not depend on n, which gives the uniformity in time. Substituting it into formula (3.63) and summing over sequences of skeletons rather than their unordered collections, we obtain

$$\begin{aligned}\sum_{\substack{\gamma\in\mathcal{K}^n_\Lambda\\ x\in\mathcal{P}(\gamma)}} \mathrm{e}^{(1+a)|\mathcal{P}(\gamma)|}\left|\bar{\varrho}^n_{\eta,\eta'}(\gamma)\right| &\leqslant \sum_{n=1}^\infty\frac{1}{(n-1)!}\sum_{\substack{\mathfrak{S}_1,\ldots,\mathfrak{S}_n\,\mathrm{cluster}\\ x\in\mathcal{P}(\mathfrak{S}_1)}}\prod_{\alpha=1}^n \mathrm{e}^{(1+a)|\mathcal{P}(\mathfrak{S}_\alpha)|}(2\tau_a)^{|\mathfrak{S}_\alpha|}\\ &\leqslant \sum_{n=1}^\infty\frac{1}{(n-1)!}\sum_{\mathcal{T}_n}\sum_{\substack{\mathfrak{S}_1,\ldots,\mathfrak{S}_n,x\in\mathcal{P}(\mathfrak{S}_1)\\ \mathcal{T}_n\subset\mathcal{H}(\mathfrak{S}_1,\ldots,\mathfrak{S}_n)}}\prod_{\alpha=1}^n\left(2\tau_a\,\mathrm{e}^{(1+a)b}\right)^{|\mathfrak{S}_\alpha|}.\end{aligned} \tag{3.65}$$

In the last expression, the second sum runs over all tree-graphs on the sequence $\{1, 2, \ldots, n\}$. We also used the estimate $|\mathcal{P}(\mathfrak{S}_\alpha)| \leqslant b|\mathfrak{S}_\alpha|$; recall that $b = |B|$ is the size of the dependence set B of the perturbation β^Λ. To finish the proof, we need the estimates contained in the following lemmas:

Lemma 3.10. *Provided that $z \geqslant 0$ and $c > 1$ are such that the inequality $(1+2cbz)^b \leqslant c$ is satisfied, one has*

$$\sum_{\mathfrak{S}:x\in\mathcal{P}(\mathfrak{S})} z^{|\mathfrak{S}|} \leqslant cbz. \tag{3.66}$$

Lemma 3.11. *If $z(\mathfrak{S}) \geqslant 0$ for all skeletons, then the inequality*

$$\sum_{\mathcal{T}_n} \sum_{\substack{\mathfrak{S}_1,\ldots,\mathfrak{S}_n;x\in\mathcal{P}(\mathfrak{S}_1)\\ \mathcal{T}_n\subset\mathcal{H}(\mathfrak{S}_1,\ldots,\mathfrak{S}_n)}} \prod_{\alpha=1}^{n} z(\mathfrak{S}_\alpha) \leqslant \frac{1}{2}(n-2)! \left[\sup_x \sum_{\mathfrak{S}:x\in\mathcal{P}(\mathfrak{S})} (2\mathrm{e})^{|\mathcal{P}(\mathfrak{S})|} z(\mathfrak{S})\right]^n \tag{3.67}$$

holds true for any $n > 1$.

Using these lemmas and a trivial estimate for $n = 1$, we immediately obtain

$$\begin{aligned}\sum_{\substack{\gamma\in\mathcal{K}_\Lambda^n\\ x\in\mathcal{P}(\gamma)}} \mathrm{e}^{(1+a)|\mathcal{P}(\gamma)|} \sup_{\eta,\eta'} \left|\bar{\varrho}^n_{\eta,\eta'}(\gamma)\right| &\leqslant \frac{1}{2}\sum_{n=1}^{\infty}\left[\sum_{\mathfrak{S}:x\in\mathcal{P}(\mathfrak{S})} \left(2^b\,\mathrm{e}^{(2+a)b}\tau_a\right)^{|\mathfrak{S}|}\right]^n \\ &\leqslant \frac{b}{2}\sum_{n=1}^{\infty}\left[2^b\mathrm{e}^{(2+a)b}cb\tau_a\right]^n \leqslant 1\end{aligned} \tag{3.68}$$

provided that the conditions

$$[1+2\tilde{\tau}]^b \leqslant c \qquad \frac{\tilde{\tau}}{1-\tilde{\tau}} \leqslant 2 \tag{3.69}$$

are satisfied with $\tilde{\tau} = 2^b\,\mathrm{e}^{(2+a)b}cb\tau_a$. Clearly, for any $a \geqslant 0$ and $c > 1$, the above conditions are fulfilled by choosing τ_a small enough, which finishes the proof of inequality (3.58).

Proof of lemma 3.10. First of all, we can write the inequalities

$$\sum_{\mathfrak{S}:x\in\mathcal{P}(\mathfrak{S})} z^{|\mathfrak{S}|} \leqslant \sum_{\mathfrak{S}:\mathcal{R}(\mathfrak{S})=\{x\}} |\mathcal{P}(\mathfrak{S})| z^{|\mathfrak{S}|} \leqslant \frac{b}{2}\sum_{\mathfrak{S}:\mathcal{R}(\mathfrak{S})=\{x\}} (2z)^{|\mathfrak{S}|} \tag{3.70}$$

where we used that

$$|\mathcal{P}(\mathfrak{S})| \leqslant b|\mathfrak{S}| \leqslant b2^{|\mathfrak{S}|-1}. \tag{3.71}$$

Recall that the root set, $\mathcal{R}(\mathfrak{S})$, of any simple polymer $\mathfrak{S}$ contains exactly one site. Let $\gamma \in \mathfrak{S}$ be any (simple) polymer from $\mathfrak{S}$ and $(x, k) \in \gamma$ be an interaction point. If $(x_1, k_1), \ldots, (x_m, k_m)$ is the (unique) sequence of interaction points from γ such that (x_i, k_i) points to (x_{i+1}, k_{i+1}) for any $i = 1, \ldots, m-1$ and $(x_1, k_1) = (x, k)$ and (x_m, k_m) is the root of γ, then we say that (x, k) is of *order m*. The order of γ is defined as the maximal order of its interaction points. It is evident that all polymers $\gamma \in \mathfrak{S}$ are of the same order, so, we can consider it as the order of the skeleton $\mathfrak{S}$. Let us define

$$\mathcal{Y}_m = \sum_{\substack{\mathfrak{S}:\mathcal{R}(\mathfrak{S})=\{x\}\\ \text{order}\leqslant m}} (2z)^{|\mathfrak{S}|}. \tag{3.72}$$

To prove the lemma, it is sufficient to show that $\mathcal{Y}_m \leqslant 2cz$ for all $m \geqslant 1$ provided that the assumption of the lemma, $(1+2cz)^b \leqslant c$, is satisfied. Proceeding by induction, let $\mathcal{Y}_p \leqslant 2cz$ for all $p < m$. Any skeleton $\mathfrak{S}$ of order m is uniquely introduced by its root and by the collection of skeletons $\{\mathfrak{S}_1, \ldots, \mathfrak{S}_r\}$ of orders $\leqslant m-1$ and with roots $x_1, \ldots, x_r$ pointing to the root x. Clearly, there are at most b possibilities for roots $x_1, \ldots, x_r$ and we can write the inequalities

$$\mathcal{Y}_m \leqslant 2z\sum_{Y\subset B}\prod_{y\in Y}\mathcal{Y}_{m-1} = 2z(1+\mathcal{Y}_{m-1})^b \leqslant 2z(1+2cz)^b \leqslant 2cz \tag{3.73}$$

proving the lemma. □

Proof of lemma 3.11. Following [12], chapter V, we distinguish tree-graphs according to the multiplicities of vertices. According to the elementary graph theory, the sequence $d_1, \ldots, d_n$ of multiplicities is an arbitrary sequence of positive integers satisfying $\sum_i d_i = 2(n-1)$. Re-arranging the terms on the left-hand side of (3.67), we can write

$$\mathrm{LHS}_{(3.67)} = \sum_{\substack{d_1,\ldots,d_n \geqslant 1 \\ \sum_i d_i = 2(n-1)}} \sum_{\mathcal{T}_n(d_1,\ldots,d_n)} w(\mathcal{T}_n \mid d_1, \ldots, d_n) \tag{3.74}$$

where the second sum runs over all tree-graphs $\mathcal{T}_n$ with the multiplicities $d_1, \ldots, d_n$ of its vertices and

$$w(\mathcal{T}_n \mid d_1, \ldots, d_n) = \sum_{\substack{\mathfrak{S}_1,\ldots,\mathfrak{S}_n; x\in\mathcal{P}(\mathfrak{S}_1) \\ \mathcal{T}_n \subset \mathcal{H}(\mathfrak{S}_1,\ldots,\mathfrak{S}_n)}} \prod_{\alpha=1}^{n} z(\mathfrak{S}_\alpha). \tag{3.75}$$

To estimate that, we proceed as follows. Since $\mathcal{T}_n$ is a tree-graph, there exists (at least one) vertex $\bar{\alpha} \neq 1$ such that $d_{\bar{\alpha}} = 1$. Without any loss of generality we can assume that $\bar{\alpha} = n$ and, moreover, that the edge $(n-1, n) \in \mathcal{T}_n$. Therefore, removing the vertex n, we obtain a tree-graph $\mathcal{T}_{n-1}$ with multiplicities $d_1, \ldots, d_{n-2}, d_{n-1} - 1$. Since by the assumption $\mathcal{P}(\mathfrak{S}_n) \cap \mathcal{P}(\mathfrak{S}_{n-1}) \neq \emptyset$ to fit $\mathcal{T}_n$, we can write the estimate

$$w(\mathcal{T}_n \mid d_1, \ldots, d_n) \leqslant \sum_{\substack{\mathfrak{S}_1,\ldots,\mathfrak{S}_{n-1}; x\in\mathcal{P}(\mathfrak{S}_1) \\ \mathcal{T}_{n-1} \subset \mathcal{H}(\mathfrak{S}_1,\ldots,\mathfrak{S}_{n-1})}} \prod_{\alpha=1}^{n-1} z(\mathfrak{S}_\alpha) |\mathcal{P}(\mathfrak{S}_{n-1})| \sup_x \sum_{\mathfrak{S}: x\in\mathcal{P}(\mathfrak{S})} z(\mathfrak{S}). \tag{3.76}$$

Iterating this process, we arrive at the inequality

$$w(\mathcal{T}_n \mid d_1, \ldots, d_n) \leqslant \left[\sum_{\mathfrak{S}: x\in\mathcal{P}(\mathfrak{S})} |\mathcal{P}(\mathfrak{S})|^{d_1} z(\mathfrak{S}) \right] \prod_{\alpha=2}^{n} \left[\sup_y \sum_{\mathfrak{S}: y\in\mathcal{P}(\mathfrak{S})} |\mathcal{P}(\mathfrak{S})|^{d_\alpha - 1} z(\mathfrak{S}) \right]. \tag{3.77}$$

Substituting it into (3.74) and by using Cayley's formula for the number of tree-graphs with fixed multiplicities of vertices,

$$\#\{\mathcal{T}_n(d_1, \ldots, d_n)\} = \frac{(n-2)!}{\prod_{\alpha=1}^{n} (d_\alpha - 1)!} \tag{3.78}$$

we immediately obtain

$$\begin{aligned} \mathrm{LHS}_{(3.74)} &\leqslant \sum_{d_1,\ldots,d_n=1}^{\infty} \frac{(n-2)!}{\prod_{\alpha=1}^{n} (d_\alpha - 1)!} \\ &\quad \times \left[|\mathcal{P}(\mathfrak{S})|^{d_1} \sum_{\mathfrak{S}: x\in\mathcal{P}(\mathfrak{S})} z(\mathfrak{S}) \right] \left[\sup_y \sum_{\mathfrak{S}: y\in\mathcal{P}(\mathfrak{S})} |\mathcal{P}(\mathfrak{S})|^{d_\alpha - 1} z(\mathfrak{S}) \right]^{n-1} \\ &= (n-2)! \left[\sum_{\mathfrak{S}: x\in\mathcal{P}(\mathfrak{S})} |\mathcal{P}(\mathfrak{S})| \, \mathrm{e}^{|\mathcal{P}(\mathfrak{S})|} z(\mathfrak{S}) \right] \left[\sup_y \sum_{\mathfrak{S}: y\in\mathcal{P}(\mathfrak{S})} \mathrm{e}^{|\mathcal{P}(\mathfrak{S})|} z(\mathfrak{S}) \right]^{n-1}. \end{aligned} \tag{3.79}$$

Now, the lemma follows by using the inequality $|\mathcal{P}(\mathfrak{S})| \leqslant 2^{|\mathcal{P}(\mathfrak{S})|-1}$. □

The case of weakly correlated initial data requires a generalization of lemma 3.9 to superpolymers, using the notation of section 3.5. Since the proof goes along the same lines, we only sketch it.

Lemma 3.12. *Given $a \geqslant 0$, there exist constants $\tau'_a, \upsilon_a > 0$ such that the following is true. If the condition*

$$\frac{\|\beta^\Lambda\|}{\tau'_a \varepsilon_0} + \sup_x \sum_{A \ni x} \mathrm{e}^{\upsilon_a |A|} \left(\mathrm{e}^{\|V(A)\|} - 1\right) \leqslant 1 \tag{3.80}$$

is satisfied, then

$$\sup_x \sup_n \sum_{\substack{\langle \Gamma, \mathfrak{A} \rangle \in \mathfrak{K}^n_\Lambda \\ x \in \mathcal{P}(\langle \Gamma, \mathfrak{A} \rangle)}} \mathrm{e}^{(1+a)|\mathcal{P}(\langle \Gamma, \mathfrak{A} \rangle)|} \sup_{\eta'} \left| w^{\Lambda,n}_{\eta'}(\langle \Gamma, \mathfrak{A} \rangle) \right| \leqslant 1. \tag{3.81}$$

Proof. Using equation (3.48) and the fact that $\nu^{\Lambda,n}_{\eta'}$ is a probabilistic measure, we can estimate the weight of a super-polymer $\langle \Gamma, \mathfrak{A} \rangle$ by

$$\left| w^{\Lambda,n}_{\eta'}(\langle \Gamma, \mathfrak{A} \rangle) \right| \leqslant \sup_\eta \prod_{\gamma \in \Gamma} \left| \varrho^{\Lambda,n}_{\eta,\eta'}(\gamma) \right| \prod_{A \in \mathfrak{A}} \left(\mathrm{e}^{\|V(A)\|} - 1\right). \tag{3.82}$$

In order to fit it to the geometrical formalism of the proof of lemma 3.9, we only need to generalize the notion of simple parts. A super-polymer $\langle \Gamma, \mathfrak{A} \rangle$ will be called simple whenever either (i) Γ is a simple polymer and $\mathfrak{A} = \emptyset$ or (ii) $|\mathfrak{A}| = 1$ and $\Gamma = \emptyset$. Every super-polymer may then be uniquely split into its simple parts. For 'polymer-like' simple parts the skeleton representation is unchanged and, for convenience, we can include the sets $A \in \mathfrak{A}$ into the family of skeletons, defining formally $\mathcal{P}(A) = A$. As a result, we can assign to any super-polymer $\langle \Gamma, \mathfrak{A} \rangle$ a family $\mathfrak{S} = \{\mathfrak{S}_\alpha\}$, where $\mathfrak{S}_\alpha$ stands either for a skeleton or for a set of sites, and with the graph of connectivity, $\mathcal{H}(\mathfrak{S})$, being connected. Repeating now the proof of lemma 3.9, we evidently arrive at the following variant of (3.65):

$$\sum_{\substack{\langle \Gamma, \mathfrak{A} \rangle \in \mathfrak{K}^n_\Lambda \\ x \in \mathcal{P}(\langle \Gamma, \mathfrak{A} \rangle)}} \mathrm{e}^{a|\mathcal{P}(\langle \Gamma, \mathfrak{A} \rangle)\|} \sup_{\eta'} \left| w^{\Lambda,n}_{\eta'}(\langle \Gamma, \mathfrak{A} \rangle) \right| \leqslant \sum_{n=1}^\infty \frac{1}{(n-1)!} \sum_{\mathcal{T}_n} \sum_{\substack{\mathfrak{S}_1, \ldots, \mathfrak{S}_n, x \in \mathcal{P}(\mathfrak{S}_1) \\ \mathcal{T}_n \subset \mathcal{H}(\mathfrak{S}_1, \ldots, \mathfrak{S}_n)}} \prod_{\alpha=1}^n u(\mathfrak{S}_\alpha) \tag{3.83}$$

where we have denoted

$$u(\mathfrak{S}_\alpha) = \begin{cases} \left(2\, \mathrm{e}^{ab \frac{\|\beta^\Lambda\|}{\varepsilon_0}}\right)^{|\mathfrak{S}_\alpha|} & \text{if } \mathfrak{S}_\alpha \text{ is a skeleton} \\ \left(\mathrm{e}^{\|V(\mathfrak{S}_\alpha)\|} - 1\right) & \text{if } \mathfrak{S}_\alpha \text{ is a set of sites.} \end{cases} \tag{3.84}$$

Applying again lemmas 3.10 and 3.11, the above expression may be further estimated by

$$\begin{aligned} &\frac{1}{2} \sum_{n=1}^\infty \left[\sup_x \sum_{\mathfrak{S}: x \in \mathcal{P}(\mathfrak{S})} (2\mathrm{e})^{\mathcal{P}(\mathfrak{S})} u(\mathfrak{S}) \right]^n \\ &\qquad \leqslant \frac{1}{2} \sum_{n=1}^\infty \left[2^{b+1} \mathrm{e}^{(2+a)b} cb \frac{\|\beta^\Lambda\|}{\varepsilon_0} + \sup_x \sum_{A \ni x} (2\mathrm{e})^{|A|} \left(\mathrm{e}^{\|V(A)\|} - 1\right) \right]^n \end{aligned} \tag{3.85}$$

from which the lemma immediately follows. □

4. Continuous time models

In this section, we study continuous time interacting particle systems, proving that all the results of the last section keep valid. We follow two different approaches. First, we treat the

continuous time limits of sequences of PCA, directly applying the above results. Second, we develop a general perturbation framework for continuous time models based on the Dyson equation which enables the results to be extended to a class of models not admitting any natural discrete time approximation.

4.1. Continuous time limit of PCA

Throughout this section, we keep all the notation from the previous section. We consider a spin–flip process on $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ with transition rates $c(x, \eta)$ for all $x \in \mathbb{Z}^d$ and $\eta \in \Omega$. They are to be interpreted as the probability rates of flipping the spin at x provided that the configuration is η.[5] Restricting only to models with bounded rates, such a process may be approximated by the discrete time PCA parametrized by $\delta > 0$ small enough which has the transition probabilities

$$p_x^\delta(a \mid \eta) = (1 - \delta c(x, \eta))\mathbb{I}_{[a=\eta(x)]} + \delta c(x, \eta)\mathbb{I}_{[a=-\eta(x)]} \tag{4.1}$$

replacing further the continuous time t with the integer $n = [t/\delta]$, see [7] for instance.

As a reference process, we take a system of uncoupled spin–flip processes with the transition rates $c^{(0)}(x, \eta) = \varepsilon_{\eta(x)}$, where $\varepsilon_\pm > 0$. After discretization (4.1) it may be cast in the formalism of section 3.2 with the stochastic matrix $\mathsf{P}^{0,\delta}$ having form (3.5), where $\varepsilon_\pm^\delta = \delta\varepsilon_\pm$. From here on the superscript δ will refer to the discrete time approximation with the parameter δ. We assume the total spin–flip process to be a perturbation of the reference process and we write the spin-flip rates in the form $c = c^{(0)} + c^{(1)}$. Substituting this decomposition into (4.1), one gets its discrete form (3.14) with the perturbation part

$$\beta_x^\delta(a \mid \eta) = \delta\left(c^{(1)}(x, \eta)\mathbb{I}_{[a=-\eta(x)]} - c^{(1)}(x, \eta)\mathbb{I}_{[a=\eta(x)]}\right). \tag{4.2}$$

It has the norm $\|\beta^\delta\| = \delta\left\|c^{(1)}\right\|$, where $\left\|c^{(1)}\right\| = \sup_{x,\eta}\left|c^{(1)}(x, \eta)\right|$.

From the δ-scaling of β^δ and $\varepsilon_\pm^\delta$ we immediately note that the condition $\left\|c^{(1)}\right\|/\varepsilon_0 \ll 1$ with $\varepsilon_0 = \min\{\varepsilon_-, \varepsilon_+\}$ characterizes the weak coupling regime for the (continuous time) spin-flip process. Indeed, it implies the inequality $\|\beta^\delta\|\big/\varepsilon_0^\delta \ll 1$ for all δ which ensures a full perturbation control of the discrete time approximating processes in the neighbourhood of $\delta = 0$. More precisely, all the statements of lemma 3.2 and thus proposition 3.5 and theorem 3.6 hold true uniformly in δ, provided that $\left\|c^{(1)}\right\| \leqslant \tau_a\varepsilon_0$.

The construction of the spin-flip process as the limit $\delta \downarrow 0$ of the approximating PCA is the subject of the following proposition, the proof of which may be found in [7].

Proposition 4.1. *For all $t \geqslant 0$, there exists the weak limit $\mu_\eta^t = \lim_{\delta\downarrow 0}\mu_\eta^{\delta,[t/\delta]}$.*

Our result is then the following:

Theorem 4.2. *Let $\left\|c^{(1)}\right\| \leqslant \tau\varepsilon_0$ with $\tau = \tau_a$, $a > 0$, being the constant from lemma 3.2. Then for any $\eta \in \Omega$ and $t > 0$ one has:*

(1) The measure μ_η^t is Gibbsian.
(2) The corresponding potential is exponentially decaying and given as $U_\eta^t(A, \eta') = \lim_{\delta\downarrow 0} U_\eta^{\delta,[t/\delta]}(A, \eta')$ with the limit taken in the sense of classes of physically equivalent potentials (it means that every limit point of the RHS gives a potential of μ_η^t).

Remark 4.3. For simplicity, we restrict here only to fixed initial data. The similar statement obviously holds also for weakly coupled initial data in the sense of section 3.5.

[5] For a more general set-up see the next section or [6].

Proof. (1) We use the notation $\sigma_y^x = -\sigma_y$ if $x = y$ and $\sigma_y^x = \sigma_y$ otherwise, and also $\mu^x(\sigma) = \mu(\sigma^x)$. Let $\sigma \in \Omega$ and $t > 0$ be fixed. Since the marginal measure $\mu_\eta^{\delta,[t/\delta]}$, $\delta > 0$ is Gibbsian with the potential $U_\eta^{\delta,[t/\delta]}$, we have

$$-\log \frac{\mathrm{d}\mu_\eta^{\delta,[t/\delta],x}}{\mathrm{d}\mu_\eta^{\delta,[t/\delta]}}(\eta') = \sum_{A\ni x} \left(U_\eta^{\delta,[t/\delta]}(A,\eta'^x) - U_\eta^{\delta,[t/\delta]}(A,\eta')\right) \quad \text{a.s.} \tag{4.3}$$

where the RHS defines a continuous version of the LHS and we use the symbol $w_\eta^{\delta,[t/\delta],x}$ for it. Splitting it into the reference and the perturbation parts, the latter may be bounded due to theorem 3.6 as

$$\left|\tilde{w}_\eta^{\delta,[t/\delta],x}(\eta')\right| \leqslant 2 \sup_\delta \sup_n \sum_{A\ni x} \sup_{\eta,\eta'} \left|\tilde{U}_\eta^{\delta,n}(A,\eta')\right| \leqslant 2. \tag{4.4}$$

As the reference part has a δ-uniform bound due to lemma 3.1, we get

$$\sup_\delta \sup_x \sup_{\eta,\eta'} \left|w_\eta^{\delta,[t/\delta],x}(\eta')\right| < \infty. \tag{4.5}$$

for all $t > 0$. Similarly,

$$\left|w_\eta^{\delta,[t/\delta],x}(\eta'_\Lambda \eta''_{\Lambda^c}) - w_\eta^{\delta,[t/\delta],x}(\eta')\right| \leqslant 4 \sup_\delta \sup_n \sum_{\substack{A\ni x\\ A\not\subset\Lambda}} \sup_{\eta,\eta'} \left|\tilde{U}_\eta^{\delta,n}(A,\eta')\right| \tag{4.6}$$

which yields

$$\lim_\Lambda \sup_\delta \sup_x \sup_{\eta,\eta',\eta''} \left|w_\eta^{\delta,[t/\delta],x}(\eta'_\Lambda \eta''_{\Lambda^c}) - w_\eta^{\delta,[t/\delta],x}(\eta')\right| = 0. \tag{4.7}$$

It follows from (4.5) and (4.7) that $\left\{w_\eta^{\delta,[t/\delta],x}(\eta')\right\}_{\delta>0}$ is a uniformly bounded equicontinuous family of functions of η'. By Ascoli's theorem, it contains a uniformly convergent subsequence (along a sequence $\delta_n \downarrow 0$) with limit $w_\eta^{t,x}(\eta')$. To finish the proof that μ_η^t is a Gibbsian measure, it suffices to show that $w_\eta^{t,x}$ defines a continuous version of $\mathrm{d}\mu_\eta^{t,x}\big/\mathrm{d}\mu_\eta^t$. Note that it also implies that the limit $w_\eta^{t,x}(\eta')$ does not depend on the subsequence. However, the above statement follows from the following simple calculation (we omit the indices η and $[t/\delta_n]$). For any local function f we can write

$$\begin{aligned}\mu^{\delta_n,x}(f) &= \mu^{\delta_n}\left(f\mathrm{e}^{-w^{\delta_n,x}}\right)\\ &= \mu^{\delta_n}\left(f\mathrm{e}^{-w^x}\right) + \mu^{\delta_n}\left(f\left[\mathrm{e}^{-w^{\delta_n,x}} - \mathrm{e}^{-w^x}\right]\right)\end{aligned}$$

and by using proposition 4.1 and the bound

$$\left|\mu^{\delta_n}\left(f\left[\mathrm{e}^{-w^{\delta_n,x}} - \mathrm{e}^{-w^x}\right]\right)\right| \leqslant \|f\|\mathrm{e}^{\|w^{\delta_n,x}\|}\left(\mathrm{e}^{\|w^{\delta_n,x}-w^x\|} - 1\right) \xrightarrow{n\to\infty} 0 \tag{4.9}$$

we immediately obtain

$$\mu^x(f) = \mu\left(f\mathrm{e}^{-w^x}\right). \tag{4.10}$$

(2) As the existence of the limit $U_\eta^{0,t}(A,\eta') = \lim_{\delta\downarrow 0} U_\eta^{0,\delta,[t/\delta]}(A,\eta')$ is obvious from lemma 3.1, it suffices to concentrate on the perturbation part of the potential. If we define the norm of any potential V by[6]

$$\|V\|_a = \sup_x \sum_{A\ni x} \mathrm{e}^{a|A|} \sup_\eta |V(A,\eta)| \tag{4.11}$$

for any $a > 0$, then $\{V; \|V\|_a \leqslant 1\}$ is a compact Banach space, see [15] for instance. Using the bound $\left\|\tilde{U}_\eta^{\delta,[t/\delta]}\right\| \leqslant 1$ for all δ, one can choose a subsequence $\delta_n \to 0$ such that there is the limit $\tilde{U}_\eta^t = \lim_n \tilde{U}_\eta^{\delta_n,[t/\delta_n]}$ in norm (4.11). Theorem 4.17 and proposition 4.19 in [4] then imply that U_η^t is the potential of the (limit) measure μ_η^t. □

[6] Note that $V(A) = 0$ whenever the set A is not connected.

4.2. General model

The configuration space of our model is $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ and its elements are denoted by $\sigma, \eta, \ldots$. Given a subset $\Lambda \subset \mathbb{Z}^d$, the symbol σ_Λ is the restriction of σ to the set Λ; the set of all configurations in Λ is denoted by Ω_Λ. A function $f : \Omega \to \mathbb{R}$ is called local if it depends only on the restrictions to a finite set $D \subset \mathbb{Z}^d$; the minimal set with this property is called the dependence set and we use the symbol $\mathcal{D}_f$ for it. The set of all local functions is denoted by $\mathcal{L}$.

First, we introduce the reference dynamics by choosing it as the system of uncoupled spin-flip processes. Its generator is $L^{(0)} = \sum_x L_x^{(0)}$, where

$$L_x^{(0)} f(\sigma) = \varepsilon_{\sigma_x}(f(\sigma^x) - f(\sigma)) \tag{4.12}$$

for any $f \in \mathcal{L}$ with spin-flip rates $0 < \varepsilon_-, \varepsilon_+ < \infty$. Let $\varepsilon_0 = \min\{\varepsilon_-, \varepsilon_+\}$ and $2\varepsilon = \varepsilon_- + \varepsilon_+$. In the above formula, $\sigma_y^x = -\sigma_y$ iff $x = y$ and it is equal to σ_y otherwise.

As the perturbation, we consider a general class of processes allowing arbitrary many-spin transformations provided they are 'local enough' and 'weak enough', see below the condition in theorem 4.4. It is introduced by a collection of transition rates $c_T^{(1)}(\sigma, \eta) \geqslant 0$ for any finite set $T \subset \mathbb{Z}^d$ and any configurations $\sigma \in \Omega$, $\eta \in \Omega_T$. They are to be interpreted as the rates at which the transition $\sigma \to \eta_T \sigma_{T^c}$ occurs. We assume there is a map $T \to \mathcal{P}(T)$, assigning to any finite set T a finite set $\mathcal{P}(T) \supset T$, such that $c_T^{(1)}(\sigma, \eta) = c_T^{(1)}(\sigma', \eta)$ whenever $\sigma_{\mathcal{P}(T)} = \sigma'_{\mathcal{P}(T)}$. The norm of any transition rate $c_T^{(1)}$ is defined by

$$\left\| c_T^{(1)} \right\| = \sup_{\sigma,\eta} \left| c_T^{(1)}(\sigma, \eta) \right|. \tag{4.13}$$

For any function $f \in \mathcal{L}$, let $L^{(1)} f = \sum_T L_T^{(1)} f$ with

$$L_T^{(1)} f(\sigma) = \sum_{\eta \in \Omega_T} c_T^{(1)}(\sigma, \eta)[f(\eta_T \sigma_{T^c}) - f(\sigma)]. \tag{4.14}$$

The whole process under study is then defined by the operator $L = L^{(0)} + L^{(1)}$ which may be extended to a generator of a Markov semigroup, provided that certain conditions on transition rates are satisfied, see [6] for details.

In place of working directly with the infinite-volume process, we construct its finite-volume approximations, proceeding along the same lines as for the PCA. Given a finite set Λ, we consider an interacting particle system on Ω_Λ defined through the generator $L^\Lambda = L^{(0)} + L^{(1),\Lambda}$, where $L^{(0)}$ is unchanged and $L^{(1),\Lambda}$ is constructed from the transition rates $c_T^{(1),\Lambda}(\sigma, \eta)$ for every $T \subset \Lambda$ and $\sigma \in \Omega_\Lambda$, $\eta \in \Omega_T$. We use $S^\Lambda(t) = \exp(tL^\Lambda)$ for the corresponding semigroup. Let $\alpha \geqslant 1$ be given. We say that the process in Λ is an *α-approximant* of the infinite-volume process iff (1) $c_T^{(1),\Lambda}(\sigma, \eta) = c_T^{(1)}(\sigma, \eta)$ whenever $\mathcal{P}(T) \subset \Lambda$ and (2) $\left\| c_T^{(1),\Lambda} \right\| \leqslant \alpha \left\| c_T^{(1)} \right\|$ for all $T \subset \Lambda$. Where it makes no confusion, we will mostly omit the superscript Λ.

Before stating our theorem, we need to introduce the following notation. A set $A \subset \mathbb{Z}^d$ is said to be *connected* if one cannot write $A = A_1 \cup A_2$ with $A_{1,2} \neq \emptyset$ and $d(A_1, A_2) > 1$. Here we use the metric $d(x, y) = \max_i |x_i - y_i|$. For any set $A \subset \mathbb{Z}^d$ we define its *connected size* by

$$|A|_{\mathrm{con}} = \inf_{\substack{A' \supset A \\ A' \text{connected}}} |A'|. \tag{4.15}$$

Theorem 4.4. *There are constants $\tau, \varsigma > 0$ such that whenever the condition*

$$\sup_x \sum_{T : x \in \mathcal{P}(T)} \frac{\mathrm{e}^{\varsigma |\mathcal{P}(T)|_{\mathrm{con}}}}{\left[\frac{\varepsilon_0}{2\varepsilon}(1 - \mathrm{e}^{-2\varepsilon t_0})\right]^{|T|-1}} \frac{\alpha \left\| c_T^{(1)} \right\|}{\tau \varepsilon_0} \leqslant 1 \tag{4.16}$$

is true for some time $t_0 \geqslant 0$, then one has the following:

(1) For every initial configuration $\sigma \in \Omega$ and time $t > t_0$, there exists a unique measure μ_σ^t such that $\lim_\Lambda \delta_{\sigma_\Lambda} S_\Lambda(t) = \mu_\sigma^t$ weakly for any sequence of α-approximants.

(2) The measure μ_σ^t is Gibbsian with a potential exponentially decaying in the connected size (4.15), uniformly in $\geqslant t_0 + \epsilon$, $\epsilon > 0$.

Remark 4.5. Our condition (4.16) is actually stronger than the condition for the uniform exponential ergodicity, see [6] for a standard argument. Using this and part 2 of the theorem, the argument in the proof of theorem 4.2 immediately gives the Gibbsianness of the invariant measure. With some additional effort, one can also prove the exponential convergence of the potential in the limit $t \uparrow \infty$.

Remark 4.6. A generalization of the theorem to allow high-temperature initial data also holds. Recalling the notation from section 3.5, condition (4.16) gets the general form

$$\sup_x \left[\sum_{T: x \in \mathcal{P}(T)} \frac{\mathrm{e}^{\varsigma |\mathcal{P}(T)|_{\mathrm{con}}}}{\left[\frac{\varepsilon_0}{2\varepsilon}(1 - \mathrm{e}^{-2\varepsilon t})\right]^{|T|-1}} \frac{\alpha \left\| c_T^{(1)} \right\|}{\tau \varepsilon_0} + \sum_{A \ni x} \mathrm{e}^{\upsilon |A|} \left(\mathrm{e}^{\|V(A)\|} - 1 \right) \right] \leqslant 1 \tag{4.17}$$

where V is the potential of the initial Gibbs measure. Since the formalism of super-polymers introduced in section 3.5 works here without any essential changes, we will only concentrate on the proof of theorem 4.4.

4.3. Dyson expansion

In what follows, let a finite volume $\Lambda \subset \mathbb{Z}^d$ and a time $t > 0$ be fixed. The semigroup $S(t)$ is the solution of the Dyson equation (sometimes also referred to as the Duhamel formula),

$$S(t) = S^{(0)}(t) + \int_0^t \mathrm{d}\tau\, S(\tau) L^{(1)} S^{(0)}(t - \tau). \tag{4.18}$$

By iterating it and splitting the generator $L^{(1)}$ into the sum of local contributions, we arrive at the Dyson series (note that there is no convergence problem for finite-dimensional operators)

$$S(t) = \sum_{n=0}^{\infty} \sum_{T_1, \ldots, T_n} \int_0^t \mathrm{d}t_1 \int_{t_1}^t \mathrm{d}t_2 \cdots \int_{t_{n-1}}^t \mathrm{d}\, t_n\, S^{(0)}(t_1) L_{T_1}^{(1)} S^{(0)}(t_2 - t_1) L_{T_2}^{(1)} \cdots L_{T_n}^{(1)} S^{(0)}(t - t_n). \tag{4.19}$$

Any finite sequence $\Gamma = [T_1, t_1; T_2, t_2; \ldots; T_n, t_n]$ will be called an *interaction set*, whenever $T_1, \ldots, T_n \subset \Lambda$ and $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t$. Assigning to it the *unnormalized weight* by[7]

$$\rho([T_1, t_1; \ldots; T_n, t_n]) = S^{(0)}(t_1) L_{T_1}^{(1)} S^{(0)}(t_2 - t_1) L_{T_2}^{(1)} \cdots L_{T_n}^{(1)} S^{(0)}(t - t_n) \tag{4.20}$$

we can formally write series (4.19) in the form

$$S(t) = \int \mathcal{D}\Gamma \rho(\Gamma) \tag{4.21}$$

where we have introduced the notation

$$\int \mathcal{D}\Gamma \equiv \sum_{n=0}^{\infty} \sum_{T_1, \ldots, T_n} \int_0^t \mathrm{d}\, t_1 \int_{t_1}^t \mathrm{d}t_2 \cdots \int_{t_{n-1}}^t \mathrm{d}t_n. \tag{4.22}$$

If it is necessary to indicate the dependence on the volume Λ and the time t, we use the extended notation for the weight, $\rho_\Lambda(\Gamma; t)$. The matrix elements of the operator $S(t)$ in the

[7] Note that the unnormalized weight is defined as an operator on $C(\Omega_\Lambda)$.

natural basis, $S_{\sigma,\eta}(t) = \delta_\sigma S(t)\mathbf{1}_\eta$, have the interpretation of the probabilities of finding the configuration η at time t, starting from the configuration σ at time zero. Here δ_σ is the measure concentrated on σ and $\mathbf{1}_\eta(\eta') = 1$ for $\eta' = \eta$ and 0 otherwise. We introduce the (*normalized*) operator $\bar{S}(t)$ by its matrix elements $\bar{S}_{\sigma,\eta}(t) = \frac{S_{\sigma,\eta}(t)}{S^{(0)}_{\sigma,\eta}(t)}$. Similarly, we assign to any interaction set Γ the (*normalized*) weight $\bar{\rho}(\Gamma)$ with $\bar{\rho}_{\sigma,\eta}(\Gamma) = \frac{\rho_{\sigma,\eta}(\Gamma)}{S^{(0)}_{\sigma,\eta}(t)}$.

Given an interaction set $\Gamma = [T_1, t_1; T_2, t_2; \ldots; T_n, t_n]$, we define its *support*, $\underline{\Gamma} = \bigcup_k T_k$, and the *projection set*, $\mathcal{P}(\Gamma) = \bigcup_k \mathcal{P}(T_k)$. We say that Γ is *connected* whenever it cannot be split into two non-empty interaction sets Γ_1 and Γ_2 such that $\mathcal{P}(\Gamma_1) \cap \mathcal{P}(\Gamma_2) \neq \emptyset$. As in the case of PCA, we divide interaction sets into 'essentially independent' parts which allow for a well-controlled cluster expansion. In order to avoid additional technicalities, we define polymers as subsets of $\mathbb{Z}^d$. We start with the observation that the normalized weight $\bar{\rho}(\Gamma)$ of any interaction set Γ factorizes into the product over all connected components of Γ. Indeed, if $\{\gamma_i\}$ is the family of connected components of Γ, then formula (4.20) reads, writing explicitly the dependence on the volume,

$$\rho_\Lambda(\Gamma) = S^{(0)}_{\Lambda\setminus\mathcal{P}(\Gamma)}(t) \otimes \bigotimes_i \rho_{\mathcal{P}(\gamma_i)}(\gamma_i). \tag{4.23}$$

It immediately follows that $\bar{\rho}(\Gamma) = \bigotimes_i \bar{\rho}(\gamma_i)$ and also $\bar{\rho}_\Lambda(\Gamma) = \bar{\rho}_{\mathcal{P}(\Gamma)} \otimes 1\!\!1_{\Lambda\setminus\mathcal{P}(\Gamma)}$. Defining for any set $M \subset \Lambda$ the (normalized) weight $\bar{\varrho}(M) = \bar{\varrho}(M; t)$ by

$$\bar{\varrho}(M; t) = \int_{\substack{\Gamma \text{ connected} \\ \mathcal{P}(\Gamma)\cap\Lambda = M}} \mathcal{D}\Gamma \bar{\rho}(\Gamma; t) \tag{4.24}$$

we can write the normalized semigroup $\bar{S}(t)$ as

$$\bar{S}(t) = \sum_{\mathcal{M}} \prod_{M\in\mathcal{M}} \bar{\varrho}(M) \tag{4.25}$$

where the sum runs over all disjoint collections of subsets of Λ. So, we have obtained the expansion of the semigroup in the form of a polymer model with the polymers being finite subsets of Λ and with the compatibility defined as the disjointness of sets. Note that the weight of the empty set has matrix elements $\bar{\varrho}_{\sigma,\eta}(\emptyset) = 1$. The cluster expansion

$$\log \bar{S}_{\sigma,\eta}(t) = \sum_{\mathcal{M}} \bar{\varrho}^T_{\sigma,\eta}(\mathcal{M}) \tag{4.26}$$

with the sum running over all clusters of polymers in Λ and $\bar{\varrho}^T(\mathcal{M})$ being the weight of the cluster $\mathcal{M}$, allows the marginal measure $\delta_\sigma S(t)$ to be written in the Gibbs form

$$(\delta_\sigma S(t))(\eta) = S^{(0)}_{\sigma,\eta}(t)\, \mathrm{e}^{-\sum_A U^t_\sigma(A,\eta)}. \tag{4.27}$$

Here, the reference semigroup $S^{(0)}(t)$ has a simple product structure and the interacting part of the potential is

$$U^t_\sigma(A, \eta) = -\sum_{\mathcal{M}:\bigcup_{M\in\mathcal{M}} M = A} \bar{\varrho}^T_{\sigma,\eta}(\mathcal{M}). \tag{4.28}$$

Note that the potential $U^t_\sigma(A)$ depends on the volume Λ as long as this Λ is still not large enough.

Defining the operator norm

$$\|\mathcal{O}\| = \sup_{\sigma,\eta} |\mathcal{O}_{\sigma,\eta}| \tag{4.29}$$

for any operator on $C(\Omega_\Lambda)$, we have the following statement about the convergence of cluster expansions:

Lemma 4.7. *Given $a \geqslant 0$, there are constants $\tau_a, \varsigma_a > 0$ such that the condition*

$$\sup_x \sum_{T:x\in\mathcal{P}(T)} \frac{\mathrm{e}^{\varsigma_a|\mathcal{P}(T)|_{\mathrm{con}}}}{\left[\frac{\varepsilon_0}{2\varepsilon}(1-\mathrm{e}^{-2\varepsilon t_0})\right]^{|T|-1}} \frac{\left\|c_T^{(1)}\right\|}{\tau_a\varepsilon_0} \leqslant 1 \tag{4.30}$$

satisfied with some $t_0 \geqslant 0$ implies the bound

$$\sup_x \sup_{t>t_0} \sum_{\mathcal{M}:x\in\cup_{M\in\mathcal{M}}M} \mathrm{e}^{a|\mathcal{M}|_{\mathrm{con}}} \|\bar{\varrho}^T(\mathcal{M};t)\| \leqslant 1 \tag{4.31}$$

where the sum runs over all clusters in the volume Λ and in the time interval $[0,t]$. We also used the notation $|\mathcal{M}|_{\mathrm{con}} = \sum_{M\in\mathcal{M}} |M|_{\mathrm{con}}$.

4.4. Sketch of proofs

The proof of theorem 4.4 is based on lemma 4.7 and both go along the same lines as in the case of PCA. That is why we only sketch the main ideas of the proof of lemma 4.7 and then we recall section 3.4.

In what follows, we use the 'canonical' notation $\Gamma = [T_1, t_1; \ldots; T_n, t_n]$. We represent it by building on Γ a directed graph $\mathcal{G}(\Gamma)$ with $k \rightsquigarrow l$ iff (1) $k < l$, (2) $T_k \cap \mathcal{P}(T_l) \neq \emptyset$, and (3) $k < k' < l \Rightarrow T_k \cap \mathcal{P}(T_k') = \emptyset$. Every vertex can only point to one another and if it points to none, than it is called a *root*. We use the notation $\bar{t}_k = t_l - t_k$ iff $k \rightsquigarrow l$ and $\bar{t}_k = t - t_k$ whenever k is a root. By repeating the construction from section 3.6, we also define *skeletons* as the natural equivalence classes of interaction sets with exactly one root.

Introducing the notation $R_T^{(0)}(t) = S_T^{(0)}(t) - S_T^{(0)}(\infty)$, we first observe that $R_T^{(0)}(t)\mathbf{1} = 0$, which implies $R_T^{(0)}(t)S_T^{(0)}(t') = R_T^{(0)}(t+t')$. Further, $L_T^{(1)}\mathbf{1} = 0$ and, as a consequence, $L_T^{(1)}S_T^{(0)}(t) = L_T^{(1)}R_T^{(0)}(t)$. By using this, the unnormalized weight (4.20) may be written in the form

$$\rho(\Gamma) = S_\Lambda^{(0)}(t_1)L_{T_1}^{(1)}R_{T_1}^{(0)}(t_2-t_1) \otimes S_{\Lambda\setminus T_1}^{(0)}(t_2-t_1)L_{T_2}^{(1)}R_{T_2}^{(0)}(t_3-t_2) \otimes S_{\Lambda\setminus T_2}^{(0)}(t_3-t_2)L_{T_3}^{(1)}\ldots \tag{4.32}$$

which may be further simplified as $k \rightsquigarrow l$ implies

$$L_{T_k}^{(1)}R_{T_k}^{(0)}(t_{k+1}-t_k)S_{T_k}^{(0)}(t_{k+2}-t_{k+1})\ldots S_{T_k}^{(0)}(t_l-t_{l-1}) = L_{T_k}^{(1)}R_{T_k}^{(0)}(t_l-t_k). \tag{4.33}$$

In words, every operator in (4.20) of the form $S_{T_k}^{(0)}(\bar{t}_k)$ may be replaced with $R_{T_k}^{(0)}(\bar{t}_k)$; compare with (3.57).

Proof of lemma 4.7. Recalling a standard argument for the convergence of cluster expansions, see [5], it suffices to prove the bound

$$\sup_x \sup_{t>t_0} \sum_{M:x\in M} \mathrm{e}^{(1+a)|M|_{\mathrm{con}}} \|\bar{\varrho}(M;t)\| \leqslant 1 \tag{4.34}$$

where the sum runs over all polymers in Λ and $[0,t]$, which further follows from the inequality

$$\sup_x \sup_{t>t_0} \int_{\substack{\Gamma \text{ connected} \\ \mathcal{P}(\Gamma)\ni x}} \mathcal{D}\Gamma\, \mathrm{e}^{(1+a)|\mathcal{P}(\Gamma)|_{\mathrm{con}}} \|\bar{\rho}(\Gamma;t)\| \leqslant 1. \tag{4.35}$$

By using (4.32) and (4.33), and the inequality $\|\mathcal{O}_\Lambda\mathcal{O}'_\Lambda\| \leqslant 2^{|\Lambda|}\|\mathcal{O}_\Lambda\|\|\mathcal{O}'_\Lambda\|$, one can estimate the unnormalized weight as

$$|[\rho(\Gamma)]_{\sigma,\eta}| \leqslant \left[S_{\Lambda\setminus\underline{\Gamma}}^{(0)}(t)\right]_{\sigma,\eta} \prod_{k\text{ root}} 2^{|T_k|}\left\|L_{T_k}^{(1)}\right\|\left\|R_{T_k}^{(0)}(\bar{t}_k)\right\| \prod_{k\text{ not root}} 2^{2|T_k|}\left\|L_{T_k}^{(1)}\right\|\left\|R_{T_k}^{(0)}(\bar{t}_k)\right\| \tag{4.36}$$

and, realizing that $\|L_T^{(1)}\| \leqslant 2^{|T|}\|c_T^{(1)}\|$, the normalized weight is bounded by

$$\|\bar{\rho}(\Gamma)\| \leqslant \prod_{k=1}^{n}\left(2^{3|T_k|}\|c_{T_k}^{(1)}\|\frac{\|R_{T_k}^{(0)}(\bar{t}_k)\|}{\inf_{\sigma,\eta}\left[S_{T_k}^{(0)}(t)\right]_{\sigma,\eta}}\right). \tag{4.37}$$

Following the strategy of section 3.6, one can sum over all interaction sets with a family of skeletons fixed to get, cf equations (3.63) and (3.64),

$$\begin{aligned}
&\int_{\substack{\Gamma \text{ connected}\\ x\in\mathcal{P}(\Gamma)}} \mathcal{D}\Gamma\, \mathrm{e}^{(1+a)|\mathcal{P}(\Gamma)|_{\mathrm{con}}}\|\bar{\rho}(\Gamma)\| \\
&\quad\leqslant \sum_{\substack{\{\mathfrak{S}_\alpha\}\text{ cluster}\\ x\in\bigcup_\alpha \mathcal{P}(\mathfrak{S}_\alpha)}} \prod_\alpha \left[\mathrm{e}^{(1+a)|\mathcal{P}(\mathfrak{S}_\alpha)|_{\mathrm{con}}} \prod_{T\in\mathfrak{S}_\alpha} 2^{3|T|}\|c_T^{(1)}\| \int_0^t \mathrm{d}\,\bar{t}\,\frac{\|R_T^{(0)}(\bar{t})\|}{\inf_{\sigma,\eta}\left[S_T^{(0)}(t)\right]_{\sigma,\eta}}\right] \\
&\quad\leqslant \sum_{\substack{\{\mathfrak{S}_\alpha\}\text{ cluster}\\ x\in\bigcup_\alpha \mathcal{P}(\mathfrak{S}_\alpha)}} \prod_\alpha \prod_{T\in\mathfrak{S}_\alpha}\left[\frac{\mathrm{e}^{\tilde{\zeta}_a|\mathcal{P}(T)|_{\mathrm{con}}}}{\left[\frac{\varepsilon_0}{2\varepsilon}(1-\mathrm{e}^{-2\varepsilon t})\right]^{|T|-1}}\frac{\|c_T^{(1)}\|}{\varepsilon_0}\right]
\end{aligned} \tag{4.38}$$

where $\tilde{\zeta}_a > 0$ is a large enough constant and the integration was carried out by using the following lemma.

Lemma 4.8. *For any finite set T and any $t > 0$ one has*

$$\int_0^t \mathrm{d}\bar{t}\,\frac{\|R_T^{(0)}(\bar{t})\|}{\inf_{\sigma,\eta}\left[S_T^{(0)}(t)\right]_{\sigma,\eta}} \leqslant \frac{2^{|T|}}{\varepsilon_0\left[\frac{\varepsilon_0}{2\varepsilon}(1-\mathrm{e}^{-2\varepsilon t})\right]^{|T|-1}}. \tag{4.39}$$

Proof. A simple calculation gives, compare equations (3.8) and (3.9),

$$S_{\{x\}}^{(0)}(\infty) = \frac{1}{2\varepsilon}\begin{pmatrix}\varepsilon_- & \varepsilon_+\\ \varepsilon_- & \varepsilon_+\end{pmatrix} \tag{4.40}$$

and

$$R_{\{x\}}^{(0)}(t) = \frac{1}{2\varepsilon}\begin{pmatrix}\varepsilon_+\mathrm{e}^{-2\varepsilon t} & -\varepsilon_+\mathrm{e}^{-2\varepsilon t}\\ -\varepsilon_-\mathrm{e}^{-2\varepsilon t} & \varepsilon_-\mathrm{e}^{-2\varepsilon t}\end{pmatrix}. \tag{4.41}$$

So, one has $\inf_{\sigma,\eta}\left[S_{\{x\}}^{(0)}(t)\right]_{\sigma,\eta} = \frac{\varepsilon_0}{2\varepsilon}(1-\mathrm{e}^{-2\varepsilon t})$ and $\|R_{\{x\}}^{(0)}(t)\| \leqslant \mathrm{e}^{-2\varepsilon t}$. Further,

$$\begin{aligned}
\|R_T^{(0)}(t)\| &= \left\|\bigotimes_{x\in T}\left[S_{\{x\}}^{(0)}(\infty) + R_{\{x\}}^{(0)}(t)\right] - \bigotimes_{x\in T} S_{\{x\}}^{(0)}(\infty)\right\| \\
&\leqslant \sum_{\emptyset\neq T'\subset T}\prod_{x\in T'}\|R_{\{x\}}^{(0)}(t)\| \prod_{y\in T\setminus T'}\|S_{\{y\}}^{(0)}(\infty)\| \\
&\leqslant 2^{|T|}\sup_{x\in T}\|R_{\{x\}}^{(0)}(t)\| \leqslant 2^{|T|}\mathrm{e}^{-2\varepsilon t}
\end{aligned} \tag{4.42}$$

and the proof is finished after integrating over time. □

The rest of the proof of inequality (4.35) only consists in geometrical estimates in the spirit of section 3.6.

Proof of theorem 4.4. (1) It follows directly from lemma 4.7 by using the methods of section 3.4.1. We only remark that the technical lemma 3.3 must be suitably changed in this case. Here, one can prove the following representation of expectations:

$$\delta_\sigma S(t) f = \sum_\eta f(\eta) S^{(0)}_{\sigma,\eta}(t) \exp\left(\sum_{\substack{\mathcal{M};\forall M\in\mathcal{M}:\\ M\cap\mathcal{D}_f\neq\emptyset}} \bar{\varrho}^T_{\sigma,\eta}(\mathcal{M}) \right) \tag{4.43}$$

where the sum runs over all clusters which contain only polymers intersecting the dependence set of f, cf equation (3.25).

(2) The proof of the Gibbsianness of the measure $\delta_\sigma S(t)$ goes along the lines of section 3.4.2. Namely, it has the potential defined by (4.27) and (4.28), the perturbation part of which satisfies

$$\sup_x \sum_{A\ni x} \mathrm{e}^{a|A|_{\mathrm{con}}} \sup_{t>t_0} \sup_\sigma \left\| U^t_\sigma(A) \right\| \leqslant 1. \tag{4.44}$$

We only add the remark that the part of the potential which corresponds to the reference process is not bounded uniformly in time in the neighbourhood of $t = 0$. □

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References

[1] Aaronson J, Gilat D and Keane M 1992 On the structure of 1-dependent Markov chains *J. Theor. Prob.* **3** 545–61
[2] Bricmont J and Kupiainen A 1995 Coupled analytic maps *Nonlinearity* **8** 379–96
[3] Chen Mu Fa 1992 *From Markov Chains to Non-Equilibrium Particle Systems* (Singapore: World Scientific)
[4] Georgii H O 1988 *Gibbs Measures and Phase Transitions* (Berlin: de Gruyter)
[5] Kotecký R and Preiss D 1986 Cluster expansions for abstract polymer models *Commun. Math. Phys.* **103** 491–8
[6] Liggett T M 1985 *Interacting Particle Systems* (Berlin: Springer)
[7] Maes C 1993 Coupling interacting particle systems *Rev. Math. Phys.* **5** 457–75
[8] Maes C 2002 Weakly Gibbsian fields, how strong? *Proc. 13th Int. Cong. of Mathematical Physics (London, 2000)* at press
[9] Maes C and Velde K V 1993 The interaction potential of the stationary measure of a high-noise spinflip process. *J. Math. Phys.* **34** 3030–8
[10] Malyshev V A and Minlos R A 1991 *Gibbs Random Fields, Cluster Expansions* (Dordrecht: Kluwer)
[11] Pirogov S A 1986 Cluster decompositions for automata systems *Probl. Pereda. Inf.* **22** 60–6
[12] Simon B 1993 *The Statistical Mechanics of Lattice Gases* (Princeton, NJ: Princeton University Press)
[13] Toom A L, Vasilyev N B, Stavskaya O N, Mityushin L G, Kurdyumov G L and Pirogov S A 1990 Discrete local Markov systems *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis* ed R L Dobrushin, V I Kryukov and A L Toom (Manchester: Manchester University Press) pp 1–182
[14] van Enter A C D, Fernández R, den Hollander F and Redig F 2001 Possible loss and recovery of Gibbsianness during the stochastic evolution of Gibbs measures *Preprint* mp_arc 01-200 (*Commun. Math. Phys.* at press)
[15] van Enter A C D, Fernández R and Sokal A D 1993 Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory *J. Stat. Phys.* **72** 879–1167
[16] van Enter A C D, Medved' I and Netočný K 2001 Chaotic size dependence in the Ising model with random boundary conditions *Preprint* http://mpej.unige.ch/mp_arc-bin/mpa?yn=01-284